MATH 3320, HOMEWORK #11

DUE FRIDAY, NOVEMBER 9

To ensure that you get full credit, be sure to *show your work* in the problems that require calculations. Very little credit is given for answers without justification. Please write in complete sentences to help us understand what you are doing.

You may collaborate with classmates in solving the problems, including the extra credit problems. If you do so, please list their names on your assignment. However, you should not consult *any* other people (except the instructors or TAs), or use online resources. (Seriously, it's very obvious to us when this occurs, and there are drastic consequences, so don't do it!) If you use results that were not proved in class, please provide your own proof.

1.

(a) (points) Let $\frac{A}{B} > 1$ be a rational number with $gcd(A, B) = 1$ and $A, B > 0$. If $\langle a_0, \ldots, a_n\rangle$ is the CF (continued fraction) of $\frac{A}{B}$, what is $CF(\frac{B}{A})$?

(b) (points) Let $\frac{p_k}{q_k}$ be the convergents of a real number $\alpha = \langle a_0, \ldots, a_n, \ldots\rangle$. Show that $q_k \geq 2^{k/2}$.

Hints: Part is true for any $\alpha > 1$. For (b), use induction. The recursion formulas are $p_{-1} = 1,\ q_{-1} = 0$, $p_0 = a_0,\ q_0 = 1$, and $p_{n+1} = a_{n+1}p_n + p_{n-1},\ q_{n+1} = a_{n+1}q_n + q_{n-1}$. Note that this is not necessarily true for p_k; it is if you assume something about a_0.

2.

(a) (10 points) Let p_k/q_k be the $k-$th convergent of $\alpha = \langle a_0, \ldots, a_n\rangle$. Prove that $p_k/p_{k-1} = \langle a_k, \ldots, a_0\rangle$ and $q_k/q_{k-1} = \langle a_{k-1}, \ldots, a_0\rangle$.

(b) (points) A certain number α has a continued fraction expansion whose first three partial quotients are $a_0 = 1,\ a_1 = 3,\ a_2 = 4, a_3 = 6$. What can you say about an upper and a lower bound for α?

Hints: You can use the formulas for the p, q involving matrices. Or you can use Euler's formulas for p, q in Davenport. For irrational numbers the CF is infinite. For rational numbers the convergent will eventually be equal to α. This affect $<$ versus $\leq$ in the various inequalities.

3.

(a) (points) Find the best approximation to $\sqrt{2}$ among all rational numbers with denominator not exceeding 985.

(b) (points) Without computing the continued fraction expansion of $\alpha = \frac{1+\sqrt{3}}{2}$, decide if 153/112 and 571/418 are consecutive convergents of α.

Hints: Use the hint in the HW itself. The relevant result is also stated as a theorem in the slides.

4. (points) Show that $\sqrt{9n^2 + 6} = \langle 3, \overline{n, 6n}\rangle$.