MATH 3320, HOMEWORK #12

DUE FRIDAY, NOVEMBER 16

To ensure that you get full credit, be sure to *show your work* in the problems that require calculations. Very little credit is given for answers without justification. Please write in complete sentences to help us understand what you are doing.

You may collaborate with classmates in solving the problems, including the extra credit problems. If you do so, please list their names on your assignment. However, you should not consult *any* other people (except the instructors or TAs), or use online resources. (Seriously, it's very obvious to us when this occurs, and there are drastic consequences, so don't do it!) If you use results that were not proved in class, please provide your own proof.

A word on notation. Davenport writes continued fractions using square brackets, but angled brackets are also common. In this homework, we'll treat them as the same, that is,

$$\alpha = [a_0, a_1, a_2, \ldots] = \langle a_0, a_1, a_2, \ldots \rangle .$$

(Note that Davenport uses $[a_0, a_1, \ldots]$'s to stand for just the *numerator* of the expansion. For this homework, we will *not* follow this convention.)

1.

(a) (10 points) Find a positive integer n that has at least three different representations as a sum of two squares $n = a^2 + b^2$, disregarding signs or the swapping of a and b (i.e. (a, b), $(-a, b)$, $(a, -b)$, (b, a), etc. are all considered *one* solution in this sense).

(b) (10 points) Prove that given any four consecutive integers, at least one is not representable as a sum of two squares.

Solution: A square $\pmod 4$ is $0, 1, 2$. A sum of two squares is $0, 1, 2$, never 3. Among four consecutive numbers, one is $3 \pmod 4$; that one cannot be a square.

A more complicated, but essentially the same solution, is if you quote the theorem that says that a number can only be a sum of squares if and only if it is of the form $2^a \prod q_i^{2e_i} \prod p_i^{e_i}$ with $q_j \equiv 3 \pmod 4$ and $p_i \equiv 1 \pmod 4$. Such number are never $4k + 3$.

2. (20 points) Find all the positive integers x and y with $y \le 1000$ such that $x^2 - y^2 + 8xy + 1 = 0$. (Hint: complete the square; use techniques related to Pell's equation.)

3.

(a) (10 points) Find the greatest common divisor of $11 + 7i$ and $5 + 3i$.

(b) (10 points) Factor $17 + 19i$ into a product of Gaussian primes in the ring $\mathbf{Z}[i]$ of Gaussian integers.

4. (20 points) Show that an odd prime $p \equiv 1, 3 \pmod 8$ if and only if p can be written as $p = x^2 + 2y^2$. (There are many ways to do this, e.g. quadratic reciprocity, continued fractions, etc.)

Solution. If $p = 2x^2 + 2y^2$, then -2 is a square $\pmod p$. This implies $p \equiv 1, 3 \pmod 8$ using Legendre symbols.

The converse is harder. If $p \equiv 1, 3 \pmod 8$, then we can find numbers such that $x^2 + 2y^2 = mp$. We want to show there is a choice of x, y such that $m = 1$. We use the method of descent. We copy the case of $x^2 + y^2 = p$. Let $-m/2 \le u, v \le m/2$ be such that $u \equiv x \pmod m$ and

$v \equiv y \pmod{m}$. Then $u^2 + 2v^2 \equiv x^2 + 2y^2 \pmod{m}$, so $u^2 + 2v^2 = rm$. On the other hand $u^2 + 2v^2 \leq (m/2)^2 + 2(m/2)^2 = 3m^2/4 < m^2$. So $r < m$. Compute $(x + i\sqrt{2}y)(u - \sqrt{2}iv) = (xu + 2yv) + \sqrt{2}i(xv - yu)$. Then $xu + 2yv \equiv x^2 + 2y^2 \equiv 0 \pmod{m}$ and $xv - yu \equiv xy - yx \equiv 0 \pmod{m}$. Let $x_1 + \sqrt{2}iy_1 = (xu + 2yv)/m + \sqrt{2}i(xv - yu)/m$. Then

$$N[(xu + yv)/m + \sqrt{2}i(xv - yu)/m] = N[(x + i\sqrt{2}y)(u - \sqrt{2}iv)]/m^2 = (pm)(rm)/(m^2) = rp,$$

and $r < m/2$. In finitely many steps you get a solution $x_n^2 + 2y_n^2 = p$. □

5. (Extra Credit) Read the article "Triangular Numbers, Gaussian Integers, and KenKen" by John Watkins:

`https://www.jstor.org/stable/10.4169/college.math.j.43.1.037`

(You will need to use your Cornell credentials through the library to access it, or just download it while on a campus network.)

For (10 points), submit the solutions to the puzzles (d), (e) (both of which are in Figure 4), and (f) (in Figure 5).