MATH 3320, HOMEWORK #14

DUE MONDAY, DECEMBER 2

To ensure that you get full credit, be sure to *show your work* in the problems that require calculations. Very little credit is given for answers without justification. Please write in complete sentences to help us understand what you are doing.

You may collaborate with classmates in solving the problems, including the extra credit problems. If you do so, please list their names on your assignment. However, you should not consult *any* other people (except the instructors or TAs), or use online resources. (Seriously, it's very obvious to us when this occurs, and there are drastic consequences, so don't do it!) If you use results that were not proved in class, please provide your own proof.

For some problems here, you may need to use a computer to do them in a reasonable amount of time.

1.

(a) (10 points) Show that $x^2 - 2y^2 = 4$ must have a solution. (Hint: Relate the solutions to the solutions of $x^2 - 2y^2 = 1$.)

(b) (10 points) Find all solutions to $x^2 - 2y^2 = 4$. You need not write them all down explicitly, a formula will suffice.

2.

(a) (10 points) A quadratic form $ax^2 + bxy + cy^2$ with $\Delta = b^2 - 4ac > 0$ is said to be **reduced** if $0 < b < \sqrt{\Delta}$ and $\sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b$. How many reduced forms of discriminant $\Delta = 8$ are there?

(b) (10 points) Show that $x^2 - 2y^2$ and $-x^2 + 2y^2$ are equivalent. (Hint: Find solutions to $x^2 - 2y^2 = 1$ and $x^2 - 2y^2 = -1$, and use them to make changes of variables.)

(c) (10 points) What conclusion can you draw from (b) about the reduced forms you found in part (a)?

3. (15 points) Show that two forms $ax^2 \pm bxy + cy^2$ satisfying $-a < b < a < c$ and $b \neq 0$ cannot be equivalent.

HINT: Show that if they were equivalent, there would exist integers (p, q) with $gcd(p, q) = 1$ such that $a = ap^2 + bpq + cq^2$. Show that, with the given inequalities, the only choice is $p = 1, q = 0$.

4.

(a) (10 points) Use operations (i) and (ii) on page 127 in Davenport to reduce the forms $(13, 36, 25)$ and $(58, 82, 29)$ to the reduced equivalent form $(1, 0, 1)$.

(b) (10 points) Use part (a) to solve $13x^2 + 36xy + 25y^2 = 17$ and $58x^2 + 82xy + 29y^2 = 13$.

HINT: An example of this type of caclculation is in the slides from November 30

5. (Extra Credit) Every matrix A in $SL_2(\mathbf{Z})$ (i.e. 2×2 matrix with integer entries and determinant 1) can be expressed as some product of its generators

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

That is, $A = S^{n_1}T^{n_2}S^{n_3}T^{n_4}\cdots$ for some $n_i \in \mathbf{Z}$. Such expressions are not necessarily unique. For example, $TST = ST^{-1}S^3$. However, if we consider

$$R = ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

then every $A \in SL_2(\mathbf{Z})$ has a unique expression in terms of S and R as follows:

$$A = (-1)^e R^{n_0} S R^{n_1} S \cdots R^{n_{\ell-1}} S R^{n_\ell}$$

where $e \in \{0, 1\}$, $n_0, n_\ell \in \{0, 1, 2\}$, and $n_i = \{1, 2\}$ for $0 < i < \ell$.

Express the following matrices A in terms of S and R as above.

(a) (2 points)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(b) (2 points)

$$A = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$$

(c) (3 points)

$$A = \begin{bmatrix} 17 & 29 \\ 7 & 12 \end{bmatrix}$$

(d) (3 points)

$$A = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$$

for $u \in \mathbf{N}$.