MATH 3320, HOMEWORK #2

DUE FRIDAY, SEPTEMBER 7

To ensure that you get full credit, be sure to *show your work* in the problems that require calculations. Very little credit is given for answers without justification. Please write in complete sentences to help us understand what you are doing.

You may collaborate with classmates in solving the problems, including the extra credit problems. If you do so, please list their names on your assignment. However, you should not consult *any* other people (except the instructors or TAs), or use online resources. (Seriously, it's very obvious to us when this occurs, and there are drastic consequences, so don't do it!) If you use results that were not proved in class, please provide your own proof.

1. (15 points) Let n be a positive integer. Let $\sigma_0(n)$ denotes the number of positive divisors of n. (This is not the usual divisor function. For example, since 6 has positive divisors 1, 2, 3, and 6, we have $\sigma_0(6) = 4$.) Prove the following identity:

$$\prod_{\substack{d|n \\ d>0}} d = n^{\sigma_0(n)/2}.$$

In other words, the product of all positive divisors of n is equal to $n^{\sigma_0(n)/2}$.

2. Let n be a positive integer. Let $\sigma_0(n)$ be as in the previous problem.

(a) (10 points) Prove that the number of positive integer solutions of $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ is $\sigma_0(n^2)$.
(b) (10 points) If n is odd, prove that the number of integer solutions of $x^2 - y^2 = n$ is $2\sigma_0(n)$.
(c) (5 points) Does (b) hold when n is even?

3.

(a) (10 points) Let a and b be relatively prime positive integers whose product ab is a perfect square. Show that a and b are both perfect squares.
(b) (10 points) Solve the Diophantine equation $x^2 = y^3 + y$.

4. (15 points) Prove that the equation $x^2 + y^2 + z^2 = 8xy$ has no nontrivial solutions. (Hint: Look at the equation modulo 4. What does that imply about x, y, and z? Show that this implies that x, y, and z must be all be infinitely divisible by some integer and so must be 0.)

5. (15 points) Find all the primitive integer solutions of $x^2+y^2 = 2z^2$. (Hint: There are at least two possible approaches. One is to mimic the proof of the theorem on primitive Pythagorean triples. Another is to prove directly that $(\frac{x+y}{2}, \frac{x-y}{2}, z)$ is a primitive Pythagorean triple.)

6. (Extra Credit) Below is a modified version of a test I made up when I was 18. It predicts the age that is best for you to get married. (Obviously, don't take this too seriously.)

Step 1. Select the number corresponding to your favorite month. (1 for January, 2 for February, ..., 12 for December)
Step 2. To this number, add your current age in years.
Step 3. Multiply this number by 3 and take the sum of the digits.
Step 4. Again, multiply the number you have by 3, and take the sum of the digits.

Step 5. Add the century that we're in now (If 1999 was the last year of the 20th century, we know that 2018 is in the...).
Step 6. From this number, subtract the number of children you would like to have.
Step 7. The resulting number is your ideal age for marriage.

For (10 points), discover, with proof, the secret behind this test.