Continued Fractions I

I tried to keep to the notation in Davenport; it is not always possible. The material is somewhat different.

Let α be any real number.

1. Write $\alpha = (a_0 = [\alpha]) + \beta_1;\ 0 \leq \beta_1 < 1.$
2. If $\beta_1 = 0$, STOP. Otherwise, write

$$\alpha = a_0 + \frac{1}{\alpha_1} = a_1 + \frac{1}{a_1 = [\alpha_1] + \beta_2}.$$

Continue indefinitely, or until the first $\beta_r = 0$.

The process produces a sequence $\{a_0, \ldots, a_r, \ldots\}$. It stops if and only if α is rational. In general it associates a sequence $\{a_0, \ldots, a_r, \ldots\}$ with a_i positive integers for $i > 0$ to each real number.

The CF expansion is UNIQUE. Exercise, see Davenport.

Continued Fractions II

Example (Euclidean Algorithm)

$$\frac{8}{3} = 2 + \frac{2}{3} = 2 + \frac{1}{\frac{3}{2}} = 2 + \frac{1}{1 + \frac{1}{2}}.$$

Definition

We denote by $\langle a_0, \ldots, a_r \rangle$ the value of the continued fraction. In the example it is $\frac{8}{3} = \langle 2, 1, 2 \rangle$. We want to know the partial results of the CF and in particular how to find a solution to $8x + 3y = 1$.

Davenport uses $q_i \leftrightarrow a_i, \quad A_i \leftrightarrow p_i, \quad B_i \leftrightarrow q_i$. The symbol $[q_0, \ldots, q_r]$ denotes the numerator of the CF in Davenport; he then writes $\frac{A_r}{B_r} = \frac{[q_0, \qquad q_{r1}]}{[q_1, \ldots, q_r]}$ for the value of the CF.

Continued Fractions III

I will write $\frac{p_r}{q_r} = \langle a_0, \ldots a_r \rangle$ for the value of the CF.

Continued Fractions IV

Theorem (recursions)

Define p_r, q_r by

$$\begin{pmatrix} p_r & q_r \\ p_{r-1} & q_{r-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \cdots \cdot \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix}$$

Equivalently, $\begin{pmatrix} p_r & p_{r-1} \\ q_r & q_{r-1} \end{pmatrix} = \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix} \cdot \cdots \cdot \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}.$

Then $p_r = p_{r-1} a_r + p_{r-2}, \; q_r = q_r a_{r-1} + q_{r-2}.$

Furthermore,

$$\alpha = \frac{p_{r-1}\alpha_r + p_{r-2}}{q_{r-1}\alpha_r + q_{r-2}}, \quad \alpha_r = \frac{p_r\alpha + p_{r-1}}{q_r\alpha + q_{r-1}}, \quad \frac{p_r}{q_r} = \langle a_0, \dots a_r \rangle.$$

The terms $\langle a_0, \dots, a_r \rangle$ are called the CONVERGENTS.

Continued Fractions V

Proof.

The following relation holds: $\langle a_0, \ldots, a_r \rangle = a_0 + \dfrac{1}{\langle a_1, \ldots, a_r \rangle}$.

Do an induction. ☐

Proposition

$$p_r q_{r-1} - q_r p_{r-1} = (-1)^{r-1},$$
$$p_r q_{r-2} - p_{r-2} q_r = (-1)^r a_r.$$

In particular, $gcd(p_r, q_r) = 1$.

Continued Fractions VI

Proof.

The determinants of any of the matrices are (-1). The other formula comes from

$$\begin{pmatrix} p_r & p_{r-1} \\ q_r & q_{r-1} \end{pmatrix} = \begin{pmatrix} a_{r-1} & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_{r-2} & p_{r-1} \\ q_{r-2} & q_{r-1} \end{pmatrix}$$

The fact that the matrices have determinant $(-1)^{r-1}$ implies that

$$\begin{pmatrix} p_{r-1} & p_{r-2} \\ q_{r-1} & q_{r-2} \end{pmatrix}^{-1} = (-1)^{r-2} \begin{pmatrix} q_{r-2} & -p_{r-2} \\ -q_{r-1} & p_{r-1} \end{pmatrix}$$

☐

Continued Fractions VII

Example. $\alpha = \frac{16}{6}$ has the same continued fraction as $\frac{8}{3}$, namely $[2, 1, 2]$. The matrix that comes out is $\begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$ If you want the *gcd*, you have to write $\frac{16}{6} = \frac{8}{3}$, and adjust.
The way the calculations are set up is as follows; set $p_{-1} = 1$, $q_{-1} = 0$ and $p_0 = a_0, q_0 = 1$. Then use the reccurence formula to compute successively:

i	-1	0	1	2
a_i	$-$	2	1	2
p_i	1	2	3	8
q_i	0	1	1	3

The determinant $(-1)^{2-1} = -1$ is in the four entries we get $8 \cdot 1 - 3 \cdot 3 = -1$. This gives the solution to $8x + 3y = 1$.

Periodic Continued Fractions I

Example. The continued fraction of $\sqrt{6}$ is

$$\sqrt{6} = 2 + (\sqrt{6} - 2) = 2 + \frac{1}{\frac{1}{\sqrt{6}-2}} = 2 + \frac{1}{\frac{\sqrt{6}+2}{2}} = 2 + \frac{1}{2 + \frac{\sqrt{6}-2}{2}} =$$

$$= 2 + \frac{1}{2 + \frac{1}{\sqrt{6}+2}} = 2 + \frac{1}{2 + \frac{1}{4+(\sqrt{6}-2)}} = \dots .$$

This implies that the CF of $\sqrt{6}$ is $\langle 2, 2, 4, 2, 4 \dots \rangle = \langle 2, \overline{2, 4} \rangle$ where the meaning of the underlined is that it repeats periodically. The table is

i	-1	0	1	2	3	4
a_i	–	2	2	4	2	4
p_i	1	2	5	22	49	218
q_i	0	1	2	9	20	89

Periodic Continued Fractions II

The entries for $i = 1$ give $5^2 - 6 \cdot 2^2 = 1$ and those for $i = 3$ also give $49^2 - 6 \cdot 20^2 = 1$. You can also verify $(5 + 2\sqrt{6})^2 = 49 + 20\sqrt{6}$, and so on. On the other hand, $22^2 - 6 \cdot 9^2 = -2$. Then $128^2 - 6 \cdot 89^2 = -2$ as well.

For $\sqrt{5}$, the pattern is $\langle 2, \overline{4} \rangle$. The table is

i	-1	0	1	2	3	4
a_i	$-$	2	4	4	4	
p_i	1	2	9	38	161	
q_i	0	1	4	17	72	

For $i = 0$ we have $2^2 - 5 \cdot 1^2 = -1$. For $i = 1$, we get $9^2 - 5 \cdot 4^2 = 1$. For $i = 2$ we verify $38^2 - 5 \cdot 17^2 = -1$, and so on.

General Pattern I

The key fact is that $\sqrt{D} = \langle a_0, \overline{a_1, \ldots, a_r, a_{r+1} = 2a_0} \rangle$, and $\sqrt{D} = \langle a_0, \ldots, a_r, \alpha_{r+1} = \sqrt{D} + a_0 \rangle$. Note that $a_0 = p_0$.

We get an identity $\sqrt{D} = \frac{p_r(\sqrt{D}+a_0)+p_{r-1}}{q_r(\sqrt{D}+a_0)+q_{r-1}}$ which yields

$$q_r\sqrt{D}(\sqrt{D} + a_0) + \sqrt{D}q_{r-1} = (\sqrt{D} + a_0)p_r + p_{r-1}.$$

This implies

$$\begin{cases} Dq_r = a_0 p_r + p_{r-1}, \\ a_0 q_r + q_{r-1} = p_r. \end{cases}$$

Plug into the relationship $p_r q_{r-1} - p_{r-1} q_r = (-1)^{r-1}$:

$$p_r(p_r - a_0 q_r) - q_r(Dq_r - a_0 p_r) = (-1)^{r-1}.$$

General Pattern II

The conclusion is $p_r^2 - Dq_r^2 = (-1)^{r-1}$. So depending on whether r is even of odd, we get a solution to $x^2 - Dy^2 = 1$ or $x^2 - Dy^2 = -1$.

In the second case, just go to the next period.

The argument can be carried out with either of $\sqrt{D} \pm a_0$.

The equation $x^2 - Dy^2 = N$ has a solution IF AND ONLY IF N comes from a convergent $\frac{p_r}{q_r}$.

Consequences of the recursion formulas I

Proposition

Write $c_i := \frac{p_i}{q_i}$. *The relations*

$$c_{i+1} - c_i = \frac{(-1)^i}{q_{i+1} q_i}, \qquad c_{i+2} - c_i = \frac{(-1)^i a_i}{q_{i+2} q_i}$$

hold. Furthermore,

$$c_{2i} = \frac{p_{2i}}{q_{2i}} > c_{2i+2} = \frac{p_{2i+2}}{q_{2i+2}}, \qquad c_{2i+1} = \frac{p_{2i+1}}{q_{2i+1}} < c_{2i-1} = \frac{p_{2i-1}}{q_{2i-1}}$$

and

$$c_{2i} < c_{2i+1}.$$

$\lim_{n\to\infty} = \alpha$ *exists, and* $CF(\alpha) = \{a_0, \ldots, a_i, \ldots\}$. *Write* $\langle a_0, \ldots, a_k, \ldots \rangle$ *for the limit.*

Consequences of the recursion formulas II

Proof.

The fact that the limit exists is the same as the proof of the *Alternating Series Test*. So we only need to prove the second part.

Note that $a_0 = c_1 < \alpha < c_2 = a_0 + \frac{1}{a_1}$. So $c_1 = [\alpha]$. Next, recall $c_k = \langle a_0, \dots, a_k \rangle = a_0 + \frac{1}{\langle a_1, \dots, a_k \rangle}$. Taking the limit, we get $\alpha = a_0 + \frac{1}{\langle a_1, \dots \rangle}$.

☐

Corollary (1)

If $\alpha = \langle a_0, \dots \rangle = \langle b_0, \dots \rangle$, then $a_i = b_i$ for all i. The value of any infinite CF is irrational.

Proof.

Consequences of the recursion formulas III

We saw in the proof of the proposition that $a_0 = [\alpha]$. So $a_0 = b_0$. As in the proof, $\langle a_1, \ldots \rangle = \langle b_1, \ldots \rangle$, and so $a_1 = b_1$. The claim follows by induction. ☐

Corollary (2)

For any irrational $\alpha = \langle a_0, \ldots, a_k, \ldots \rangle$,

$$\alpha - c_k = \frac{(-1)^k}{q_k(\alpha_{k+1} q_k + q_{k-1})}$$

A simpler version is

$$\mid \alpha - c_k \mid < \frac{1}{q_k q_{k+1}}.$$

Consequences of the recursion formulas IV

For example, the first few convergents of π are $3, 22/7, 333/106, 355/113.$
You can check from $\pi = 3.14159\dots$ that the values alternate larger and smaller. But also $\mid \pi - \frac{22}{7} \mid < \frac{1}{742}$ and $\mid \pi - \frac{333}{106} \mid < \frac{1}{106 \cdot 113}$. You can also tell that the error, if you approximate π by 3, is less than $1/7 \approx .142$

Rational Approximation I

Theorem

Let $\alpha = \langle a_0, \ldots, a_k, \ldots \rangle$ *be irrational. Then*

$$\mid \alpha - \frac{p_{k+1}}{q_{k+1}} \mid < \mid \alpha - \frac{p_k}{q_k} \mid$$

or a sharper version

$$|q_{k+1}\alpha - p_{k+1}| < |q_k\alpha - p_k|$$

This says that the convergents are getting successively closer to α.

Rational Approximation II

Theorem

If $| \alpha - \frac{c}{d} | < | \alpha - \frac{p_k}{q_k} |$ *for some* $k \geq 1$*, then* $d > q_k$.

This says that the $k-$th convergent $\frac{p_k}{q_k}$ of the irrational α is the best approximation among all the rational numbers with denominators less than or equal to q_k.

If α is rational, the convergents stabilize to α. So you have to be careful about strict inequalitites.

Periodic Fractions I

Definition

The infinite simple fraction $\langle c_0, \dots, c_n, \overline{a_0, \dots, a_n} \rangle$ is called periodic. $\langle \overline{a_0, \dots, a_n} \rangle$ is called purely periodic.

Definition

A real number α is called quadratic irrational if it is the root of a polynomial $ax^2 + bx + c = 0$ with integer coefficients.

You need $b^2 - 4ac \geq 0$ for this.

Periodic Fractions II

Theorem

Any quadratic irrational can be written as $\alpha = \frac{R+\sqrt{D}}{S}$ *such that* $S \mid (D - R^2)$.

Proof.

The irrational satisfies an equation $ax^2 + bx + c = 0$ with $b^2 - 4ac$ not a perfect square. The solutions $\alpha_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ satisfy the requrements with $R = \pm b, S = 2a,\ D = b^2 - 4ac$. □

Any $\frac{r+\sqrt{D}}{s}$ with $s \nmid D - r^2$ can be rewritten this way; find the quadratic equation for which it is a root. For example $\frac{5+\sqrt{15}}{6} = \frac{15+\sqrt{135}}{18}$.

Periodic Fractions III

Algorithm for finding the CF of a quadratic irrational

- $a_0 = [\alpha]$. Let $\alpha_0 = (r_0 + \sqrt{D})/s_0$ as in the Theorem.
- Define r_k and s_k recursively as

  $r_{k+1} = a_k s_k - r_k, \ s_{k+1} = \frac{(D - r_{k+1}^2)}{s_k}$.

  Then $a_{k+1} = [\frac{(r_{k+1} + \sqrt{D})}{s_{k+1}}]$.

Recall $\alpha_{k+1} = \frac{1}{\alpha_k - a_k}$. Also the computation simplifies to $a_k = [\frac{r_k + a_0}{s_k}]$

The formula is $\alpha_k = \frac{r_k + \sqrt{D}}{s_k}$.

Periodic Fractions IV

Theorem (Lagrange)

$CF(\alpha)$ is periodic if and only if α is quadratic irrational. It is purely periodic if and only if α is reduced. This means $\alpha = \frac{(r+\sqrt{D})}{s} > 1$ and $\alpha' := \frac{(r-\sqrt{D})}{s}$ satisfies $-1 < \alpha' < 0$. Particular continued fractions are

$$\alpha = \sqrt{D} + [\sqrt{D}] = \langle \overline{a_0, \ldots, a_n} \rangle$$

and

$$-1/\alpha' = \sqrt{D} - [\sqrt{D}] = \langle \overline{a_n, \ldots, a_0} \rangle.$$

$$\sqrt{D} = \langle a_0, \overline{a_1, \ldots, a_n, a_{n+1} = 2a_0} \rangle.$$