Periodic Continued Fractions I

Theorem

Any quadratic irrational can be written as $\alpha = \frac{R+\sqrt{D}}{S}$ *such that* $S \mid (D - R^2)$.

Proof.

The irrational satisfies an equation $ax^2 + bx + c = 0$ with $b^2 - 4ac$ not a perfect square. The solutions $\alpha_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ satisfy the requrements with $R = \pm b, S = 2a,\ D = b^2 - 4ac$. ☐

Any $\frac{r+\sqrt{D}}{s}$ with $s \nmid D - r^2$ can be rewritten this way; find the quadratic equation for which it is a root. For example $\frac{5+\sqrt{15}}{6} = \frac{15+\sqrt{135}}{18}$.

Periodic Continued Fractions II

Algorithm for finding the CF of a quadratic irrational

Let $\alpha_0 = (r_0 + \sqrt{D})/s_0$ as in the Theorem.

- Define $a_0 = [\alpha_0]$.
- Define r_k and s_k recursively as $r_{k+1} = a_k s_k - r_k, \; s_{k+1} = \frac{D - r_{k+1}^2}{s_k}$.
- Define $a_{k+1} = [\frac{r_{k+1} + \sqrt{D}}{s_{k+1}}]$.

Proposition

- *r_k and s_k are integers, and $s_k \neq 0$.*
- *$s_k \mid (D - r_k^2)$.*
- *$\alpha_k = \frac{r_k + \sqrt{D}}{s_k}$ and therefore $a_k = [\alpha_k]$.*

Periodic Continued Fractions III

Proof.

Do an induction. The step $n = 0$ is by definition. The highlights of the induction step are:

- $r_{k+1} = a_k s_k - r_k$ implies $r_{k+1} \equiv -r_k \ (mod \ s_k)$
- So $D - r_k^2 \equiv D - r_{k+1}^2 \ (mod \ s_k)$ which implies s_{k+1} is an integer, and $\neq 0$ because $D - r_{k+1}^2$ cannot be zero.
- $\alpha_{k+1} = \frac{1}{\alpha_k - a_k} = \frac{s_k}{r_k + \sqrt{D} - a_k s_k} = \frac{s_k}{-r_{k+1} + \sqrt{D}} = \frac{r_{k+1} + \sqrt{D}}{s_{k+1}}$.

☐

NOTE: $a_k = [\frac{r_k + a_0}{s_k}]$; you can dispense with the $\sqrt{D}$.

Theorem (Lagrange)

$CF(\alpha)$ is periodic if and only if α is quadratic irrational. It is purely periodic if and only if α is reduced. This means $\alpha = \frac{(r+\sqrt{D})}{s} > 1$ and $\alpha' := \frac{(r-\sqrt{D})}{s}$ satisfies $-1 < \alpha' < 0$. Particular continued fractions are

$$\alpha = \sqrt{D} + [\sqrt{D}] = \langle \overline{a_0, \ldots, a_n} \rangle$$

and

$$-1/\alpha' = \sqrt{D} - [\sqrt{D}] = \langle \overline{a_n, \ldots, a_0} \rangle.$$

$$\sqrt{D} = \langle a_0, \overline{a_1, \ldots, a_n, a_{n+1} = 2a_0} \rangle.$$

Proposition

If the continued fraction of α is periodic, then α is quadratic irrational.

Proof.

We must have $\alpha = \langle c_0, \ldots, c_n, x\rangle$, and then $x = \langle a_0, \ldots, a_k, x\rangle$. Then $\alpha = \frac{xp_n + p_{n-1}}{xq_n + q_{n-1}}$, and $x = \frac{p_k x + p_{k-1}}{xq_k + q_{k-1}}$. The second equation implies that x is quadratic irrational; multiply out. Then the first equation implies α is quadratic irrational by the same type of argument. ☐

Example: Find the quadratic irrational corresponding to $\langle 2, \overline{1, 1, 3}\rangle$.

First find the fraction of $\langle \overline{1,1,3} \rangle$:

i	-1	0	1	2
a	$-$	1	1	3
p	1	1	2	7
q	0	1	1	4

The equation is $x = \frac{7x+2}{4x+1}$. $4x^2 - 6x - 2 = 0$. So $x = \frac{3+\sqrt{17}}{4}$.
The equation for α is $\alpha = \frac{2x+1}{x+0} = \frac{1+\sqrt{17}}{2}$.

CF of a Quadratic Irrational I

Proposition

Any quadratic irrational has a periodic CF.

Proof.

CF of a Quadratic Irrational II

This follows from the fact that $-\sqrt{D} < r_k < \sqrt{D}$ and $0 < s_k < 2\sqrt{D}$ for large k. There are only finitely many choices for r_k, s_k, so the α_m must repeat.

- $\alpha - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(\alpha_{k+1}q_k - q_{k-1})}$. Then $\left(\alpha - \frac{p_k}{q_k}\right)\left(\alpha' - \frac{p_k}{q_k}\right)$ equals $\frac{1}{q_k^2(\alpha_{k+1}q_k + q_{k-1})(\alpha'_{k+1}q_k + q_{k-1})}$.
- There is m such that $\alpha'_m < 0$. This follows because the first product on the left alternates sign, and converges to α. It follows that $\alpha'_{k+1}q_k + q_{k-1}$ must be negative.
- $-1 < \alpha'_n < 0$ for $n \geq m = k + 1$. It is enough to show this for $m + 1 : -1 < \alpha_{m+1} = \frac{1}{\alpha'_m - a_m} < 0$. This follows from $\alpha_{m+1} = \frac{1}{\alpha_m - a_m}$ and $0 < \alpha_m - a_m < 1$, and taking the conjugate.
- $\alpha_k - \alpha'_k = \frac{2\sqrt{D}}{s_k}$ and $\alpha_k \alpha'_k = \frac{r^2 - D}{s_k^2}$.

Purely Periodic CF I

Definition

A quadratic irrational $\alpha = \frac{r+\sqrt{D}}{s}$ is called reduced, if $-1 < \alpha' < 0$.

Theorem (Galois)

$CF(\alpha)$ is purely periodic if and only if α is reduced. In this case, $\alpha = \langle \overline{a_0, \ldots, a_{m-1}} \rangle$ and $\alpha' = \langle \overline{a_{m-1}, \ldots, a_0} \rangle$.