

# Bertrand's paradox (probability)

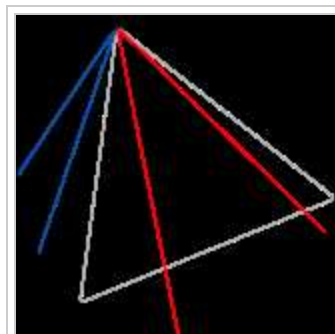
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*For other paradoxes by Joseph Bertrand, see Bertrand's paradox*

**Bertrand's paradox** is a problem within the classical interpretation of probability theory. Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?

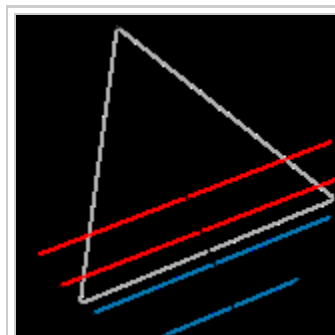
This problem was originally posed by Joseph Bertrand in his work, *Calcul des probabilités* (1888). Bertrand gave three arguments, all apparently valid, yet yielding inconsistent results.

1. The "random endpoints" method: Choose a point on the circumference and rotate the triangle so that the point is at one vertex. Choose another point on the circle and draw the chord joining it to the first point. For points on the arc between the endpoints of the side opposite the first point, the chord is longer than a side of the triangle. The length of the arc is one third of the circumference of the circle, therefore the probability a random chord is longer than a side of the inscribed triangle is one third.



Random chords,  
selection method 1;  
red = longer than  
triangle side, blue =  
shorter

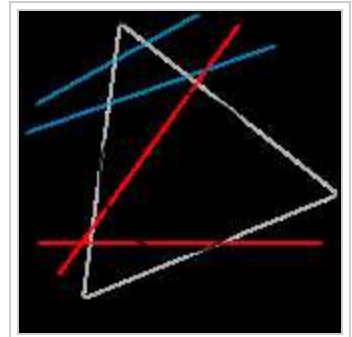
2. The "random radius" method: Choose a radius of the circle and rotate the triangle so a side is perpendicular to the radius. Choose a point on the radius and construct the chord whose midpoint is the chosen point. The chord is longer than a side of the triangle if the chosen point is nearer the center of the circle than the point where the side of the triangle intersects the radius. Since the side of the triangle bisects the radius, it is equally probable that the chosen point is nearer or farther. Therefore the probability a random chord is longer than a side of the inscribed triangle is one half.



Random chords,  
selection method 2

- 3.

The "random midpoint" method: Choose a point anywhere within the circle and construct a chord with the chosen point as its midpoint. The chord is longer than a side of the inscribed triangle if the chosen point falls within a concentric circle of radius  $1/2$ . The area of the smaller circle is one fourth the area of the larger circle, therefore the probability a random chord is longer than a side of the inscribed triangle is one fourth.



Random chords,  
selection method 3

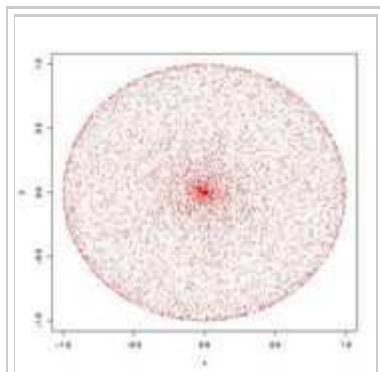
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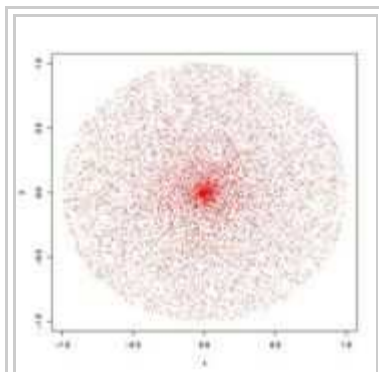
## Classical solution

The problem's solution thus hinges on the means by which a chord is chosen "at random". It turns out that once the method of random selection is specified, the problem has a well-defined solution. There is no unique selection method, so there cannot be a unique solution. The three solutions presented by Bertrand correspond to different selection methods, and in the absence of further information there is no reason to prefer one over another.

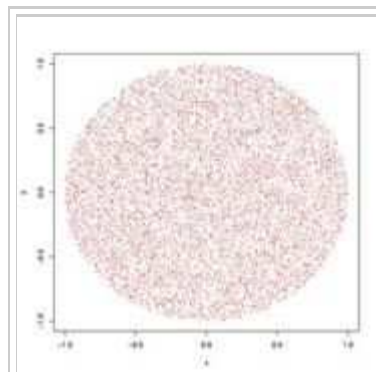
The selection methods can be visualized as follows. Other than a diameter, a chord is uniquely identified by its midpoint. Each of the three selection methods presented above yields a different distribution of midpoints. Methods 1 and 2 yield two different nonuniform distributions, while method 3 yields a uniform distribution. Other distributions can easily be imagined, many of which will yield a different proportion of chords which are longer than a side of the inscribed triangle.



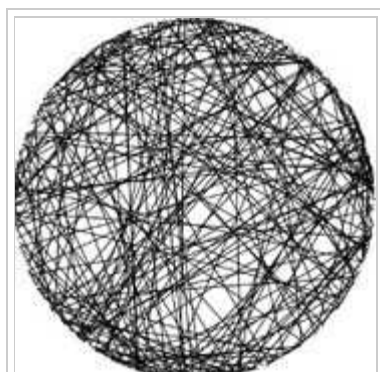
Midpoints of chords  
chosen at random,  
method 1



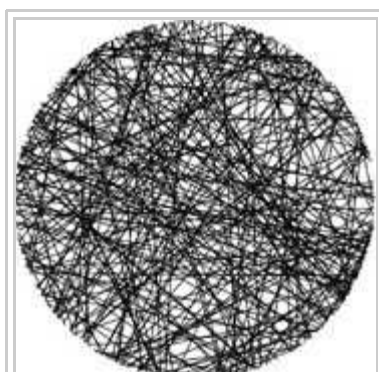
Midpoints of chords  
chosen at random,  
method 2



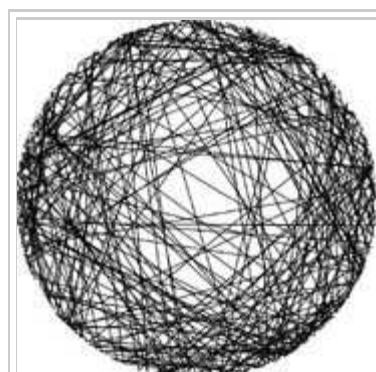
Midpoints of chords  
chosen at random,  
method 3



Chords chosen at  
random, method 1



Chords chosen at  
random, method 2



Chords chosen at  
random, method 3

## Common-sense solution using transformation groups

Bertrand's "paradox" is a paradox because the word "random" in *suppose a chord of the circle is chosen at random* is ambiguous. Similar ambiguity is found, for example, in some gas physics calculations, a field in which the correct solution can be determined by experiment. In the field of gas physics, the correct solution can be found mathematically by applying transformation groups. In his 1973 paper *The Well-Posed Problem* [1] (<http://bayes.wustl.edu/etj/articles/well.pdf>) , E. T. Jaynes used the same approach to Bertrand's paradox.

Jaynes changed the question from "what is the mathematically correct experiment" to "which properties would an acceptable physical experiment have." An experiment one could imagine is to throw straws onto a coin that lies far enough away from us such that it is sufficiently random how the straws fall onto that coin and onto the floor around the coin. The circumference of the coin represents the circle. The solution to Bertrand's paradox then is the conditional probability that the chord made by the straw is longer than the side of the inscribed triangle, given that the straw intersects the circle.

If you picture a floor full of straws thrown onto it, and a coin lying below them, it is then

common sense that the solution is the same no matter whether we take a smaller coin or a larger coin (the problem is scale invariant), and no matter whether the coin is placed a bit more to the left or a bit more to the right (the problem is translational invariant). With these two invariants, the random distribution of the chords can be calculated directly with integral equations, and the unique solution is precisely what was called "method 2" above, the *random radius* method.

This means, for the classical solutions above, if you take the chords in the pictures of method 1 or 3 above, and lay on them a smaller circle, and move that smaller circle around within the larger circle, then the probability distribution of the chords intersecting the smaller circle changes with the radius and the position of that smaller circle. In the case of "method 2", the probability distribution is always the same.

While it is not *the* mathematically correct solution, "method 2" is the only solution that fulfills the transformation invariants that practical physical experiments would have. As Jaynes pointed out in his paper, the main question is whether one should think of Bertrand's paradox as a physical experiment. Nobody said it should be one, and if one does not think of it as physical, then the "common-sense" solution described here does not apply.

## References

- Michael Clarke. *Paradoxes from A to Z*. London: Routledge, 2002.
- E. T. Jaynes. *The Well-Posed Problem*. Foundations of Physics, vol. 3, 1973, pp. 477-493. [2] (<http://bayes.wustl.edu/etj/articles/well.pdf>)

## External links

- Bertrand's Paradox (<http://www.cut-the-knot.org/bertrand.shtml>) at cut-the-knot

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