MATH 401 Talk: The Simplex Method

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1 What is Linear Programming?

1.1 Definitions

Linear Programming is about maximizing or minimizing a linear function subject to linear constraints. *Linear* means $\sum_{i=1}^{n} c_i x_i$.

Parts of a linear program:

1. *Objective function*: the linear function that you want to maximize or minimize.

2. *Constraints* of the form of $\sum_{i=1}^{n} c_i x_i$ on the left hand side and $\leq$, $=$, or $\geq$ a constant on the right hand side.

A *feasible solution* is a set of values for the x_i that satisfy the constraints.

A linear program with no feasible solutions is *infeasible*.

An *optimal* solution is a feasible solution that maximizes/minimizes the objective function. **Note:** a linear program may have more than one optimal solution, or none at all.

We can express a linear program more succinctly as:

$$\begin{aligned}
\max \quad & \vec{c} \bullet \vec{x} \\
\text{subject to} \quad & A\vec{x} \leq \vec{b} \\
& \vec{x} \geq 0 \quad [x_i \geq 0 \ \forall i]
\end{aligned}$$

It is in standard inequality form, or if it had $=$ instead of $\leq$, it would be in standard equality form.

1.2 First Example

Consider the linear program in Bland's article in the *Scientific American*:
  Let x_i = the amount of barrels of product i to produce:
i = 1: ale, i = 2: beer

$$\begin{aligned}
\text{maximize profit} \quad z &= 13x_1 + 23x_2 \\
\text{subject to} \quad 5x_1 + 15x_2 &\leq 480 \quad \text{[Limit on Corn]} \\
4x_1 + 4x_2 &\leq 160 \quad \text{[Limit on Hops]} \\
35x_1 + 20x_2 &\leq 1190 \quad \text{[Limit on Malt]} \\
x_1, \quad x_2 &\geq 0
\end{aligned}$$

The same linear program, but using matrices and vectors:

$$\text{maximize profit} \quad z = \begin{pmatrix} 13, & 23 \end{pmatrix}^T \bullet \begin{pmatrix} x_1, & x_2 \end{pmatrix}^T$$

$$\text{subject to} \quad \begin{pmatrix} 5 & 15 \\ 4 & 4 \\ 35 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\vec{x} \geq \vec{0}$$

If we simplify the constraints (divide by a positive number on both sides), the linear program becomes:

$$\begin{aligned}
\text{maximize profit} \quad z &= 13x_1 + 23x_2 \\
\text{subject to} \quad x_1 + 3x_2 &\leq 96 \quad \text{[Limit on Corn]} \\
x_1 + x_2 &\leq 40 \quad \text{[Limit on Hops]} \\
7x_1 + 4x_2 &\leq 238 \quad \text{[Limit on Malt]} \\
x_1, \quad x_2 &\geq 0
\end{aligned}$$

and

$$\text{maximize profit} \quad z = \begin{pmatrix} 13, & 23 \end{pmatrix}^T \bullet \begin{pmatrix} x_1, & x_2 \end{pmatrix}^T$$

$$\text{subject to} \quad \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 96 \\ 40 \\ 238 \end{pmatrix}$$

$$\vec{x} \geq \vec{0}$$

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

The feasible region of this linear program is a polygon.

In general, the feasible region of any linear program is a polytope (basically an n-dimensional polygon). Also, all feasible regions of linear programs are *convex*.

Definition 1 (Convexity). *A set S is* convex *if and only if for every x, $y \in S$, then the line segment $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$ is in S.*

Proposition 1. *The feasible reagion of any linear program is convex.*

Proof. I prove the case when the linear program is in standard inequality form. The same proof applies to any feasible reagion of a linear program.

Let $\vec{x}$ and $\vec{y}$ be any two points in the feasible region of the linear program. Let $\lambda \in [0, 1]$. Then we have that

$$A\vec{x} \leq \vec{b} \qquad \text{and} \qquad A\vec{y} \leq \vec{b}$$
$$\Rightarrow \quad \lambda A\vec{x} \leq \lambda\vec{b} \qquad \text{and} \qquad (1 - \lambda)A\vec{y} \leq (1 - \lambda)\vec{b}$$

$$\Rightarrow \quad \lambda A\vec{x} + (1 - \lambda)A\vec{y} \leq \lambda\vec{b} + (1 - \lambda)\vec{b}$$
$$\Rightarrow \quad A(\lambda\vec{x} + (1 - \lambda)\vec{y}) \leq \vec{b} \quad \text{[Linearity of matrix multiplication]}$$

Therefore $\lambda\vec{x} + (1 - \lambda)\vec{y}$ is in the feasile region of the linear program and the feasible region is convex. $\qquad \square$

2 Solving a Linear Program

One method of solving a linear program is the Simplex Method. The method was created by George Dantzig in 1947 (Source: Wikipedia).

First, some of definitions:

Definition 2 (Basis). *A* basis *is an index set B such that the columns of the matrix A corresponding to the indices in B is invertible. We call the corresponding matrix A_B.*

Example:

$$\text{If} \quad A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

$$\text{and} \quad B = \{1, 3\}$$

$$\text{Then} \quad A_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Definition 3 (Basic Feasible Solution). *A basic feasible solution to $A\vec{x} = \vec{b}$, $\vec{x} \geq \vec{0}$ corresponding to the basis B is a feasible solution where $x_i = 0$ if $i \notin B$.*

Example: A basic feasible solution of

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

corresponding to $B = \{1, 3\}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Fact 1. *Basic feasible solutions corresponding to a basis exist and are unique.*

Let us use the simplex method on the linear program from Bland's article. First, we convert the linear program to standard equality form. We do this by adding *slack variables*, one per constraint.

$$\begin{aligned}
\text{maximize profit} \quad & z = 13x_1 + 23x_2 \\
\text{subject to} \quad & x_1 + 3x_2 + x_3 = 96 \quad \text{[Limit on Corn]} \\
& x_1 + x_2 + x_4 = 40 \quad \text{[Limit on Hops]} \\
& 7x_1 + 4x_2 + x_5 = 238 \quad \text{[Limit on Malt]} \\
& x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0
\end{aligned}$$

To start the simplex method, we assume that we have a feasible solution to the linear program.

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In this case, we see immediately that $\vec{x} = \begin{pmatrix} 0 & 0 & 96 & 40 & 238 \end{pmatrix}^T$ is a basic feasible solution to this linear program corresponding to $B = \{3, 4, 5\}$. The objective function has value 0.

We will write the constraints in this form:

$$z - 13x_1 - 23x_2 = 0$$
$$x_1 + 3x_2 + x_3 = 96$$
$$x_1 + x_2 + x_4 = 40$$
$$7x_1 + 4x_2 + x_5 = 238$$

where $x_i \geq 0 \ \forall i$. This is the *tableau* corresponding to the basis $B = \{3, 4, 5\}$

The simplex method now solves the linear program by looking for ways to improve the solution, while keeping all of the x_i nonnegative. The simplex method works in the following way:

1. *Check for improvements.* We look at the coefficients of the X_i in the first row. If any are negative, then we know that we can improve the solution, otherwise, we have an optimal solution to the linear program.

2. *Choose an entering variable.* Pick one of the variables whose coefficients is negative in the first row to enter the basis.

3. *Ratio test.* Say we picked j to enter to the basis. To increase the objective function, we want to increase x_j as much as possible. We can increase x_j as much as the smallest positive number of (number on right hand side)/(coefficient of x_j over all of the rows but the first row). If all of these coefficients of the x_j are negative, then we can increase x_j as much as we want: then the linear program is *unbounded*. That is, we can find a feasible solution with an arbitrarily large objective value.

4. *Choose leaving variable.* We pick any variable that achieved the minimum positive ratio in the ratio test to leave the basis.

5. *Pivot on the leaving variable.* Use row operations to change the tableau so each basis variable is in exactly one row of the tableau, and no basis variables are in the first row of the tableau.

6. Now repeat the above steps until the linear program is determined to be unbounded or until an optimal solution is found.

Note: The simplex method only examines basic feasible solutions, which are extreme points (corners) of the feasible region. Because, the feasible region is convex and the objective function is linear, any local maximum of the feasible region is a global maximum. This means that it is ok for the simplex method just to examine the extreme points of the feasible reagion.

There are times when he have a choice in deciding which variable enters/leaves the basis. Such as in the beginning of the example. There are various rules for choosing which variables to pivot on, such as choose the variable with the largest coefficient in the first row of the tableau to leave/enter, or choose the one with the smallest subscript.

For the tableau above, if x_1 enters the basis, then it takes 3 pivots to solve the linear program, and if x_2 enters the basis, then it takes 2 pivots to solve the linear program.

In both cases, using the simplex method yields the final tableau

$$z + 5x_3 + 8x_4 = 800$$

$$x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = 28$$

$$x_1 - \frac{1}{2}x_3 + \frac{3}{2}x_4 = 12$$

$$-\frac{3}{2}x_3 - \frac{17}{2}x_4 + x_5 = 42$$

with basis $B = \{1, 2, 5\}$ and optimal solution $\vec{x} = \begin{pmatrix} 12 & 28 & 0 & 0 & 42 \end{pmatrix}^T$

There are no other times in this example where we have to choice between variables that enter/leave the basis.

3   Examples of Linear Programs

3.1   Kinds of Linear Programs

Fact 2 (Fundamental Theorem of Linear Programming). *Every linear program is either infeasible, has an optimal solution, or is unbounded.*

The example we used is an example of a linear program with an optimal solution.

An example of an infeasible linear program is:

$$\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 10 \\
& x_1 + x_2 = 11 \\
& x_1, \quad x_2 \geq 0
\end{aligned}$$

An example of an unbounded linear program is:

$$\begin{aligned}
\text{maximize} \quad & x_1 \\
\text{subject to} \quad & x_1 - x_2 \geq 0 \\
& x_1, \quad x_2 \geq 0
\end{aligned}$$

3.2 Running time of the simplex method

In practice, the simplex method runs very fast. However the simplex method is not a polynomial time algorithm. An example of where the simplex method requres exponentially many pivots is:

$$\begin{aligned}
\text{maximize} \quad & 4x_1 + 2x_2 + x_3 \\
\text{subject to} \quad & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& x_1, \quad x_2, \quad x_3 \geq 0
\end{aligned}$$

(From http://glossary.computing.society.informs.org/notes/Klee-Minty.pdf)

This example requires 7 pivots, when using the pivot rule to choose the entering/leaving variable to be the one whose coefficient in the first row of the tableau has the largest absolute value.

An example of a program that cycles (the same basis is reached more than once, which would cause the pivot sequence to cycle) is:

$$\begin{aligned}
\text{maximize} \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\text{subject to} \quad & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0 \\
& 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0 \\
& 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \leq 0 \\
& x_1 \leq 1 \\
& x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0
\end{aligned}$$

when the pivot rules are:

- Choose the entering variable to be the one whose coefficient in the first row of the tableau has the largest absolute value

- Choose the leaving variable to be the one with the smallest subscript

(Source: ORIE 320 course notes 9/29/05, a provided .pdf file)

Robert Bland proved that the simplex method never cycles when the pivot rule is to choose the entering variable with the smallest subscript to enter and to leave the basis whenever possible. Therefore, the simplex method always terminates.

In general, examples that take a long time for the simplex method to run or that cause the simplex method to cycle are rare. In practice, the simplex method works fast.

Although the simplex method is not a polynomial time algorithm, there are algorithms to solve linear programming in polynomial time, such as the ellipsoid method.

4 Extras

There are probably many computer packages that can solve linear programs. AMPL (Algebraic Mathematical Programming Language) is one of them.

4.1 Finding a feasible solution to a linear program

Sometimes it is difficult to find a feasible solution to a linear program. Consider the following example:

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & x_1 + x_2 = 10 \\
& x_1 - x_2 \leq -5 \\
& x_1, \quad x_2 \geq 0
\end{aligned}
$$

Now add a slack variable:

$$\text{maximize} \quad x_1 + 2x_2$$
$$\text{subject to} \quad x_1 + x_2 = 10$$
$$x_1 - x_2 + x_3 = -5$$
$$x_1, \quad x_2, \quad x_3 \geq 0$$

Modify to:

$$\text{maximize} \quad x_1 + 2x_2$$
$$\text{subject to} \quad x_1 + x_2 = 10$$
$$-x_1 + x_2 - x_3 = 5$$
$$x_1, \quad x_2, \quad x_3 \geq 0$$

To find a basic feasible solution to this problem, we introduce artificial variables, u_1 and u_2. If we can find a solution to the equations where $u_1 = u_2 = 0$, then we have a feasible solution to the linear program.

We do this by using the simplex method to solve the linear program

$$\text{maximize} \quad -u_1 - u_2$$
$$\text{subject to} \quad x_1 + x_2 + u_1 = 10$$
$$-x_1 + x_2 - x_3 + u_2 = 5$$
$$x_1, \quad x_2, \quad x_3, \quad u_1, \quad u_2 \geq 0$$

Where it is easy to identify an initial basis B, which is the basis containing the two artifical variables, and a basic feasible solution to this linear program is $x_1 = x_2 = x_3 = 0$, $u_1 = 10$, and $u_2 = 5$. Now, if the optimal solution to this linear program has objective value 0, then the optimal solution to this linear program is a feasible solution to the original program. Otherwise, the original linear program has no feasible solution.

Because the simplex method only modified the tableaus using row reductions, we can apply the same row operations to the tableaus of the original linear program. If we remove the artifical variables from the tableau of the above linear program (this linear program is feasible), we get a feasible tableau with which to start the linear program.

In this case, the simplex method yields the final tableau:

$$z + 4x_1 + u_1 + u_2 = 0$$
$$2x_1 + x_3 + u_1 - u_2 = 5$$
$$x_1 + x_2 + u_1 = 10$$

with the indicies of x_2 and x_3 as the basis B and optimal solution $x_1 = 0$, $x_2 = 10$, $x_3 = 5$, $u_1 = 0$, and $u_2 = 0$. We use this information to get:

$$z - x_1 - 2x_2 = 0$$
$$2x_1 + x_3 = 5$$
$$x_1 + x_2 = 10$$

which becomes a feasible tableau for the linear program when appropriate row operations are used:

$$z + x_1 = 20$$
$$2x_1 + x_3 = 5$$
$$x_1 + x_2 = 10$$

with basis $B = \{2, 3\}$ and optimal solution $x_1 = 0$, $x_2 = 10$, and $x_3 = 5$, with an objective value of 20.

4.2   Proving that a feasible solution is optimal

When the simplex method solved the linear program:

$$\begin{aligned}
\text{maximize profit} \quad & z = 13x_1 + 23x_2 \\
\text{subject to} \quad & 5x_1 + 15x_2 \leq 480 \quad \text{[Limit on Corn]} \\
& 4x_1 + 4x_2 \leq 160 \quad \text{[Limit on Hops]} \\
& 35x_1 + 20x_2 \leq 1190 \quad \text{[Limit on Malt]} \\
& x_1, \quad x_2 \geq 0
\end{aligned}$$

How do we know that the solution $x_1 = 12$ and $x_2 = 28$ is an optimal solution?

You can check that this solution is feasible and has objective value 800.

To prove that it is optimal, we find an upper bound on the objective function. Here is how, take the first constraint and add 2 times the second constraint to it. We get:

$$13x_1 + 23x_2 \leq 800$$

So we have found an upper bound on our objective function. Because we already have a feasible solution with objective value of 800, we know that this upper bound it tight, and therefore that our feasible solution is optimal.

The information found from the final tableau from using the simplex method can be used to generate these numbers.

5 Sources

The following sources were used for this talk:

Optimization 1 (ORIE 320) lecture notes for the Fall 2005 course

"The Allocation of Resources by Linear Programming" by Robert Bland in the June 1981 issue of *Scientific American*, volume 244, number 6, pages 126-144.

"Klee-Minty Polytope Shows Exponential Time Complexity of Simplex Method" by Harvey J. Greenberg. Found on the website http://glossary.computing.society.informs.org/notes/Klee-Minty.pdf on January 29, 2007.

The Wikipedia article "Simplex Algorithm." Found at http://en.wikipedia.org/wiki/Simplex_method.