Math. 420, Fall 2005
Final Examination
Distributed December 2, due December 12, at noon

Please remember that NEATNESS COUNTS. A solution that I can't read is automatically WRONG.

Problem 1. Consider the differential equation

$$x' = x^2 - e^{-t}. \tag{1}$$

a) What kinds of fences are the curves of equation

$$x = 0,\ x = 1,\ x = -1,\ x = e^{-t/2},\ x = -e^{-t/2}?$$

b) Solve the equation $x' = x^2 - 1$. What relation is there between these solutions and equation (1) in the half-plane $t \geq 0$?

c) Can there be any solutions $v(t)$ of equation (1), defined for $t \geq t_0$, for some t_0, such that

$$\lim_{t\to\infty} v(t) = a > 0\ ?$$

(Compare $\lim_{t\to\infty} v'(t)$ and $\lim_{t\to\infty}(v(t))^2 - e^{-t}$.)

d) Show that equation (1) has a unique solution $u(t)$ defined for all $t \in \mathbb{R}$, such that $u(t) > 0$ for all t.

e) Are there other solutions of (1) that are defined for all $t \in \mathbb{R}$.

Problem 2. a) State the fundamental inequality.

(b) Change the differential equation $x''+ax'+\sin(x) = \cos(t)$ into a system of first-order equations for

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}.$$

Find a global Lipschitz constant for this system of differential equations.

c) Show that for all values of a, all solutions of the differential equation

$$x'' + ax' + \sin(x) = \cos(t)$$

are defined for all $t \in \mathbb{R}$. Be careful to state exactly what results you are using.

d) Let $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ be a solution of the equation in part (b). If $a > 0$ (positive friction), find C in terms of a such that if $|v(t_0)| < C$ for some t_0, then $|v(t)| < C$ for all $t \geq t_0$. Hint: consider

$$\frac{d}{dt}(v)^2.$$

e) Find a step h_0 (depending on $\epsilon > 0$) such that the Euler approximation $\mathbf{u}_h$ to the solution of the differential equation of part (b) with initial condition

$$\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

has error at most ϵ for $h < h_0$ and $|t| < 1$.

Problem 3. a) Let A be a square $n \times n$ matrix with real entries. Define e^{tA} and show that $t \mapsto e^{tA}\mathbf{x}_0$ is a solution of $\mathbf{x}' = A\mathbf{x}$.

b) Find the general solution of

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(c) Describe geometrically the solutions of the equation (the word "rotation" should appear somewhere in your description).

Problem 4. Consider the system of differential equations

$$\begin{aligned} x' &= y - x^2 - a \\ y' &= y - bx + 1. \end{aligned} \tag{2}$$

a) For the values $a = -2, b = 0$, draw the curves where the vector field is horizontal and vertical.

b) If your drawing was accurate, it should allow you to find some solutions of the equation explicitly.

c) Find the zeroes of the vector field, and classify them. Sketch the trajectories of the differential equation.

Now we will vary the parameters in equation (2)

d) In the (a, b)-plane, draw the curve where the number of zeroes changes.

e) On the same drawing, draw the curve where the differential equation has a center.

f) Indicate, in the complements of the curves of parts (d) and (e), how many sinks, sources or saddles the diferential equation has.

Problem 5. a) Define the α and the ω-limit set of a bounded solution to a differential equation on $\mathbb{R}^n$.

b) describe the limit sets for all solutions of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -\sin x \end{pmatrix}.$$

c) State the Poincaré-Bendixon theorem.

d) Show that the differential equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -y + .5\sin(x) - .1x \\ x + .5\cos(y) - .1y \end{pmatrix}$$

has a limit cycle, and find a bound for points of this limit cycle.