3. The axioms of projective geometry

After being introduced to the language of the line at infinity and its points, we realize that this "extension" of the Euclidean plane has some very simple properties. We formalize these as "axioms" again, but for a different non-Euclidean geometry which call the projective plane. So we say that a projective plane is any set that consists of two kinds of elements which are called points and lines that satisfy the following three axioms, where we have a symmetric binary relation called *incidence* between points and lines (For each point and each line, they are either incident or not incident, but not both):

Axiom 1: For every pair of distinct points there is a unique line incident to both.

Axiom 2: For every pair of distinct lines there is a unique point incident to both.

Axiom 3: There are four distinct points, where no three are incident to any line.

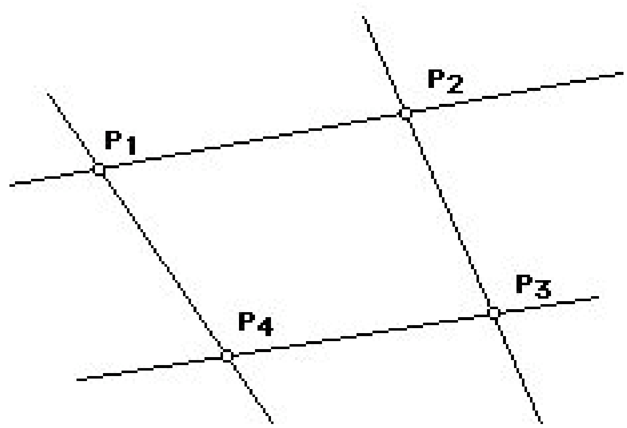

Figure 1: This shows Axiom 3 in the Euclidean plane (and the real projective plane).

Axiom 3 is included to get rid of certain unwanted degenerate examples such as indicated in Figure 2.

Note that the Euclidean plane with points at infinity and the line at infinity (the extended Euclidean plane) satisfies the axioms for a projective plane. Indeed, there are many other interesting examples, we call them models, of projective planes. For instance, there are many finite projective planes, that is projective planes with a finite number of points and lines. For example, Figure 3 indicates a finite projective plane with seven points and seven lines. Of course the points do not really have to be in the Euclidean plane. We have just used the

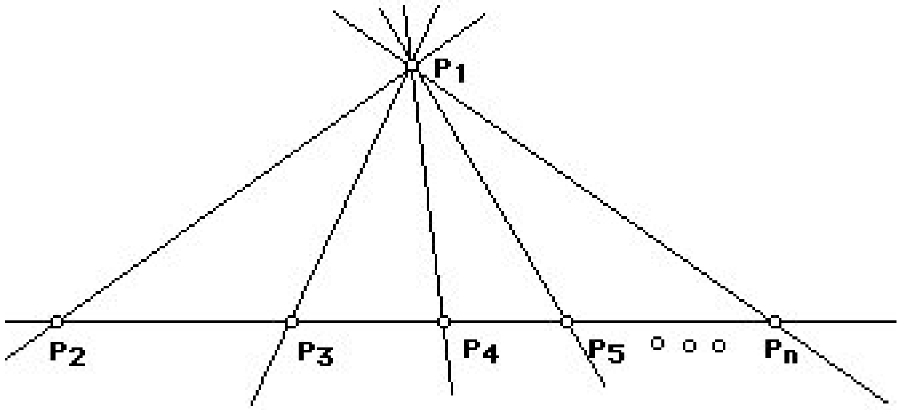

Figure 2: This indicates an example of a system of points that satisfy Axioms 1 and 2, but not 3.

picture to help indicate which points are incident to which lines. For example, $\mathbf{p}_1$, $\mathbf{p}_7$, and $\mathbf{p}_3$ are all the points incident to one line. The circle is meant to indicate that the points $\mathbf{p}_5$, $\mathbf{p}_6$, and $\mathbf{p}_7$ are also the points that are incident to one of the lines. All the other lines are arranged so that the incident points are on a straight line in the Euclidean plane.

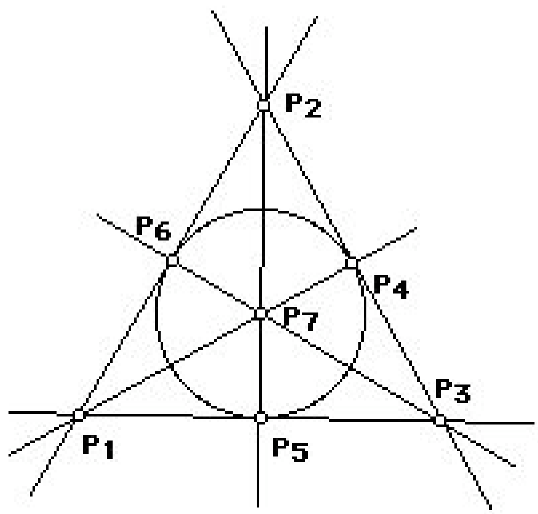

Figure 3: This shows the smallest possible projective plane with 7 points and 7 lines, the Fano plane.

1 Exercises:

In the following, assume that we have a projective plane.

1. Show that if one interchanges the words “line” and “point” in Axiom 3, then the

statement is true. Specifically, show that there are four distinct lines, no three incident with a point.

2. Given two distinct lines, show that there is a point that is not incident with either of them.

3. Let $\mathbf{p}$ be a point and L be a line in our projective plane, where $\mathbf{p}$ is not incident to L. Show that there is a one-to-one correspondence between the points incident to L and the lines incident to $\mathbf{p}$.

4. Suppose that L_1 and L_2 are two lines in our projective plane and $\mathbf{p}$ is a point not incident to either line. Show that there is a "natural" correspondence, called projection from $\mathbf{p}$, between the points incident to L_1 and the points incident to L_2. The only thing to do here is to describe projection from $\mathbf{p}$ totally in terms of incidence structure in our axioms of a projective plane.

5. Suppose that our projective plane has a finite number of lines. Suppose also that a point $\mathbf{p}$ has $n+1$ lines incident to it, and L is a line not incident to $\mathbf{p}$. Show that L is incident to exactly $n+1$ points. (Hint: Use a correspondence similar to the one in exercise 4.)

6. Suppose that our projective plane has a finite number of lines.

   (a) Show that there is a number $n \geq 2$ such that each point is incident with $n+1$ lines, and each line is incident with $n+1$ points. (Hint: Use exercises 4 and 5.)

   (b) Show that the plane has n^2+n+1 points and n^2+n+1 lines. We say that n is the *order* of the finite projective plane.) Hint: Choose some arbitrary but fixed point $\mathbf{p}$. Partition the points of the projective plane into sets, where each set consists of the points on a line incident to $\mathbf{p}$, except for $\mathbf{p}$, and the set consisting of only $\mathbf{p}$. Count the number of points in each of these sets and the number of these sets using the exercises above.