Royal Institution Mathematics Masterclass

Geometry and Perspective

E.C. Zeeman, FRS.

This is the book accompanying the video.

The Royal Institution,
21 Albemarle Street,
London W1X 4BS.

Mathematics Institute,
University of Warwick,
Coventry CV4 7AL.

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To my children.

Acknowledgements.

I first presented these ideas as part of the Royal Institution Christmas Lectures shown on BBC 2 Television in 1978, and then developed the material for use in the Mathematics Masterclasses for Young People that grew out of those lectures. I should particularly like to thank Sir George Porter, who was the Director of the Royal Institution, for inviting me to give the Christmas Lectures and for then thinking up the idea of the masterclasses and giving them his continual encouragement and support. I should also like to thank the many young people who came to those classes and made them memorable for me with their infectious enthusiasm. I am indebted to the DES for providing the financial support to make the video, and for distributing it to schools. I am grateful to John Jaworski and the BBC Open University Production Unit for making the video. I should like to thank Elaine Shiels for typing the book, and Jean-Pierre Sharp for allowing me to use his drawings of the paintings.

Preface.

In 1978, Professor Christopher Zeeman gave the Royal Institution Christmas Lectures for young people on "Mathematics into Pictures". This was the first time that mathematics had been the subject of the famous series started by Michael Faraday in 1826; it had perhaps been thought that mathematics did not lend itself to the experimental demonstrations which were such a feature of these lectures.

Professor Zeeman completely disproved this view; his six lectures were televised and were received with enthusiasm and admiration by the young audience as well as their parents, teachers and professional mathematicians. There was a clamour for more.

The Royal Institution responded by starting a series of ten "Mathematics Masterclasses" on Saturday mornings addressed to 13 year olds from schools within travelling distance of London. They were a unique combination of brief talks and demonstrations, interspersed with informal problem sessions with teachers in attendance to help when called upon. These were again a great success and similar masterclasses are now running regularly at 18 centres in Britain as well as at the Royal Institution.

Still the demand was not satisfied; the classes were oversubscribed, those who had attended wanted more, and only about sixty boys and girls could be accommodated in any one series. The importance of the masterclasses in the context of the desperate shortage of good teachers of mathematics in the schools was recognised by the Department of Education and Science who decided that the next step was to make them available to all, through video recording, and this programme on "Geometry and Perspective" is the result. It is, I hope, only the beginning.

The course is a wonderful combination of the arts and the sciences, quite demanding mathematically but absorbing and entertaining as well. We can all take part by solving the puzzles and proving the theorems that he presents in this book; it contains plenty of follow-up material along with complete solutions suitable both for teachers and for young mathematicians to study for themselves. There are few mathematicians as gifted in presenting the subject as Christopher Zeeman and we are now all able to share his brilliant lectures and learn from him something of the beauty of mathematics.

George Porter
March 1987

Introduction.

Geometry is fundamental to mathematics, and essential to the understanding of how mathematics is used in modern science. In fact to a mathematician geometrical intuition is as important as numerical skill. Nowadays there is plenty of opportunity to develop numerical skill, what with calculators, computers and statistics, but geometry unfortunately tends to be neglected in our schools. That is why geometry was chosen to be the topic of this first video of a mathematics masterclass.

Geometry provides the ideal introduction to the concepts of theorem and proof, which lie at the very heart of mathematics, as Euclid realised 2000 years ago. These concepts highlight the difference between mathematics and science. Although mathematics and science are so closely interwoven, there is all the difference in the world between a mathematical theorem and a scientific experiment, between mathematical proof and scientific proof, and it is very important to be aware of this difference.

It is equally important to appreciate the applicability of geometry. Geometry is the branch of mathematics that arises from our visual perception, which is the strongest of all our senses, and there is no more immediate application of geometry than to the study of visual perception. That is why perspective was chosen to be the second topic of this video.

The rules of perspective explain how to draw pictures and how to paint the real world on a flat canvas. The underlying theorems in geometry explain why those rules work. The proofs involve three-dimensional geometry which is more exciting and challenging that two-dimensional geometry. And the theorems have the power not only to surprise the intellect but also to astonish the eyes. They explain why some paintings look right and others look wrong. They open our eyes afresh to great works of art, and help us to draw more accurately ourselves.

I hope this video and book may appeal to people of all ages, but I particularly address myself to young students. If you are about 12 or 13 years old you will have reached the point when you can think abstractly, and can begin to appreciate geometry in all its clarity and beauty.

How to use this book.

The video is made in three parts:

Part 1 : Vanishing points (25 minutes).

Part 2 : Cubes and observation points (20 minutes).

Part 3 : Viewing distances and Brunelleschi's experiment (15 minutes).

The book contains worksheets to do after seeing each part of the video, and before seeing the next part. If you can tackle and solve all the problems on each worksheet you will not only have a better understanding of the theory, but you will also be able to appreciate the next part of the video much more deeply. The last worksheet provides an opportunity to reinforce what has been learnt, and to have a go at some harder problems. There are also summaries and mathematical notes on each part of the video, and solutions to the problems on the worksheets at the end of the book.

If you are a teacher using the video to run a class I recommend showing one part at a time. After each part photocopy the mathematical notes and worksheets, and hand then out as exercises for your students. When they have had a chance to try the problems you can help them with the ones they have not been able to solve, using the solutions. You may not want to photocopy the solutions, because you may prefer to adapt them to your own particular style of teaching, or to convert them into a language that your own class may be more used to.

If you are a young student using the video and book for private study I urge you to follow the same pattern (and not to cheat!). You will get far more out of it if you do things in the right order. Discipline yourself to switch off the video after watching Part 1, and have a go at Worksheet 1 before watching Part 2, because this will enable you to rediscover some of the mathematics for yourself before being shown it in Part 2. Mathematical discovery is not only a richly rewarding experience in itself, but it also has another more important property: anything you discover for yourself you will never forget, whereas things that other people tell you tend to go in one ear and out the other. Of course you can always look up the answers to the problems in the solutions at the end of the book, but I urge you not to do so before first having a really good try at solving them, otherwise you will not only deny yourself the excitment of discovery and the exhilaration of achievement but you may also deny yourself the gift of being able to remember it.

Geometry and perspective

Video Part 1 : Vanishing points.

When we say a painting in in perspective we mean two things: firstly it looks realistic, and secondly it obeys certain mathematical rules. These rules were originally discovered by the ancient Greeks and Romans, but were evidently forgotten again until the time of the Renaissance. They were then rediscovered by the Italian architect Filippo Brunelleschi (1377-1446) in Florence in about 1420. He explained his discovery to his fellow artists by means of an ingenious experiment with mirrors, which you will see at the end of the video, and which is explained in Mathematical Notes 3.

The first rule of perspective is that if you want to paint parallel lines going away from you then you must paint them so that they all converge towards a point, which is called a vanishing point. The underlying mathematical reason for this is explained in the video, and in Mathematical Notes 1 (page 19). The video then goes on to show several examples of paintings, some with vanishing points and others without, which are illustrated on the following pages, and which we shall now describe.

(i) <u>de Hoogh</u> : <u>*An interior scene*</u> (page 21).

This is a 17th century Dutch painting, painted about 200 years after Brunelleschi's discovery, by which time the rules of perspective were very well known. The parallel lines along the floor tiles, window panes and roof timbers all converge towards a central vanishing point; consequently the painting looks as realistic as a photograph. We shall be using this painting again in the Video Part 3 and in Worksheet 3.

(ii) <u>Hall of Kings, Alhambra</u> (page 11).

This is a Spanish wall painting, painted around 1350, about 70 years <u>before</u> Brunelleschi's discovery. Here there is no attempt to make parallel lines converge, as can be seen from the three extended lines in the lower sketch. Consequently the whole castle looks wonky.

Not that the artist would necessarily have worried about this, because he wanted to superimpose several different views at once. For example the view obtained by looking up at the ramparts as you approach the castle walls is superimposed on that obtained by looking across at the turrets from on top of the walls. The main point, however, is that the artist would not have been able to paint the castle in perspective even if he had wanted to, because the rules had not yet been rediscovered.

(iii) <u>**Roman wall painting**</u> (page 12).

This is a painting on a wall of a Roman villa dating from the Augustan period (27BC-AD14), and unearthed by archaeologists as recently as 1961. It was probably the work of a modest interior decorator, but the precision of the central vanishing point indicates clearly that the rules of perspective were known to the Romans.

I am sorry that we were unable to include this painting in the video, but it was one of the things that had to be cut to fit the video into one hour's length, for ease of copying by schools. If you want to read more about it see the Encyclopedia of World Art.

(iv) <u>Masaccio</u> : <u>*Trinity*</u> (page 13).

Masaccio (1401-1428) was a friend of Brunelleschi and must have been only about 19 when the latter discovered the rule about vanishing points. He must have been very excited when Brunelleschi explained it to him, and immediately began to use them in his own painting.

Masaccio was one of the great founders of the Renaissance, and although he died when he was only 26 he left behind a series of masterpieces that have profoundly influenced artists ever since. He painted this fresco in the church of Santa Maria Novella in Florence in 1427, a year before his death. It is a large painting on a huge flat wall, and Masaccio cleverly placed the vanishing point at eye level, so that the observer has the illusion of looking up into a little side chapel. The illusion is enhanced by the painting of an altar below, using the same vanishing point (not shown on our sketch).

(v) **Masaccio** : *The tribute money* (page 14).

This is another famous fresco of Masaccio painted the same year as the last one, this one in the Brancacci Chapel in Florence. Many people think it is his greatest work. It tells the biblical story of Jesus and his disciples entering the town of Capernaum. The tax collector, who is the man in the centre with his back to us, is demanding an entrance tax from Jesus. So Jesus turns to Peter and tells him to go and catch a fish (on the far left), and in its mouth he will find a coin with which he can then pay the taxman (on the far right).

This painting contains a wealth of innovations, not only in the colours, and in the shading of light and dark with all the figures lit from the right and in shadow from the left, but also in the psychological intensity of the expressions - real people caught in the grip of real emotions. Our particular interest, however, is in the accuracy of the perspective of the building; moreover Masaccio has cleverly placed the vanishing point at Jesus's head in order to draw the eye subconsciously towards the most important person in the picture.

You will also notice another rule of perspective: when a group of figures are level with the observer then all their heads will be roughly on the same horizontal line as the eye, while their feet will occur at different levels in the picture, those in the foreground lower and those in the background higher. This was the first picture in the world ever to obey this rule.

(vi) **Gozzoli** : *The abduction of Helen by Paris* (page 15).

This is a painting in the National Gallery in London. It shows Paris, who is standing facing us on the left with a feather in his hat, abducting Helen, who is being carried piggy-back out of her house by one of his servants, to take her to Troy. It was this event that started the Trojan war. The painting was probably painted by Gozzoli in Florence about 20 years after Masaccio. He copied Masaccio's idea of focusing attention on the most important person by placing the vanishing point at Helen's head.

If you go and look closely at the painting in the National Gallery you can see underneath the paint the original guide lines drawn by the artist extending beyond the buildings towards the vanishing point.

(vii) Veneziano : *Annunciation* (page 16).

This is a beautiful little painting by Domenico Veneziano (active 1438-1461) in the Fitzwilliam Museum in Cambridge. He painted it in Florence a few years after Masaccio's death. It shows the angel announcing to Mary that she is going to have a child. The symmetry of the pictue and the precision of the vanishing point emphasise the harmoniousness of the scene. We shall use this painting again in Mathematical Notes 3 and Worksheet 3.

(viii) Veneziano : *Miracle of St. Zenobius* (page 17).

This is another striking painting by Veneziano, originally done at the same time as the previous one, as part of the same altarpiece, and now hanging beside it in the same museum. It shows St. Zenobius about to perform the miracle of resurrecting the dead child of the kneeling mother in the centre.

An interesting feature of this painting is that Veneziano has rejected the opportunity of using a vanishing point for the parallel lines of the street. He has evidently broken the rules of perspective deliberately in order to emphasise the disharmony of the scene, in contrast to the perfect harmony of the previous painting. The scarf tossed by the mother shaking her head in anguish communicates that anguish to the observer, and the confusion of the perspective around the scarf subconsciously reinforces the emotion. A great artist knows exactly when to use the rules and when to break them.

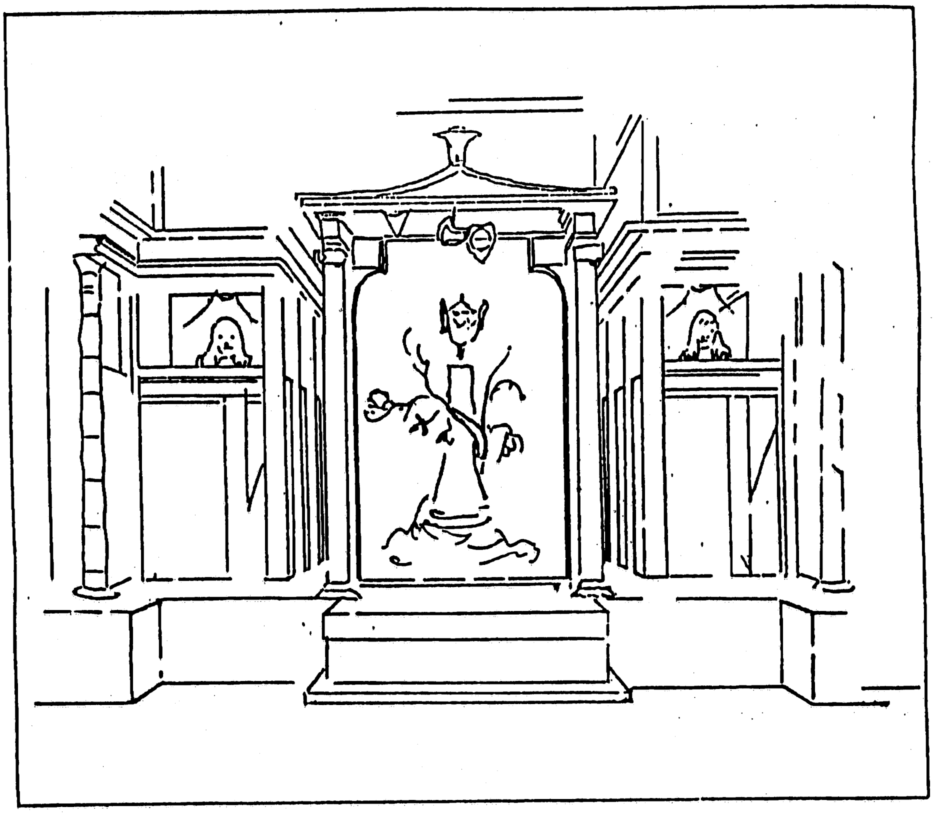
(ix) Van Eyck : *Marriage of Arnolfini* (page 18).

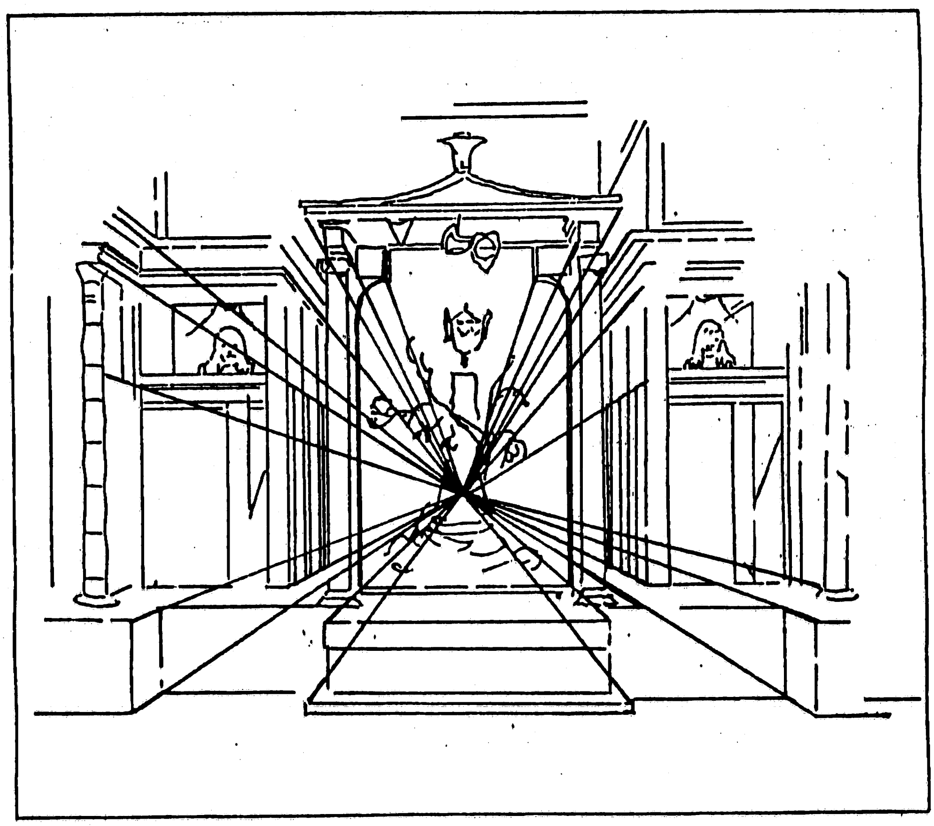
This is another well known painting from the National Gallery in London, painted by Van Eyck (active 1422-1441) in Bruges in Belgium. He painted it in 1434 just six years after Masaccio's death. He was one of the greatest living artists of his day, famous for his realism, and at first glance it looks as if the painting in in perspective. Drawing in the lines, however, shows that it is not: the floorboards, window panes and roof timbers converge to three different points. It is clear that the news had not yet reached Bruges, and that Van Eyck did not know about the uniqueness of the vanishing point. Otherwise he would have placed it at the centre of the convex mirror where he had already painted his own reflection, because he was fond of such tricks. As it is, the more you look at this painting the more you become visually aware that the perspective is not quite right.

Summarising.

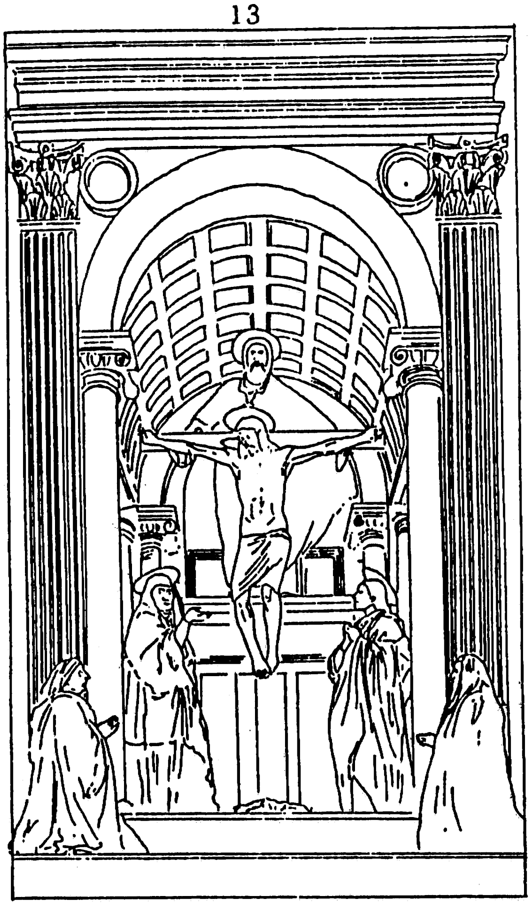
The first rule of perspective is to use vanishing points. The paintings that we have seen show how important it is to obey this rule if you want your picture to look realistic. The video and the Mathematical Notes 1 below explain why this rule works, by stating precisely and proving rigorously the underlying geometrical theorem.

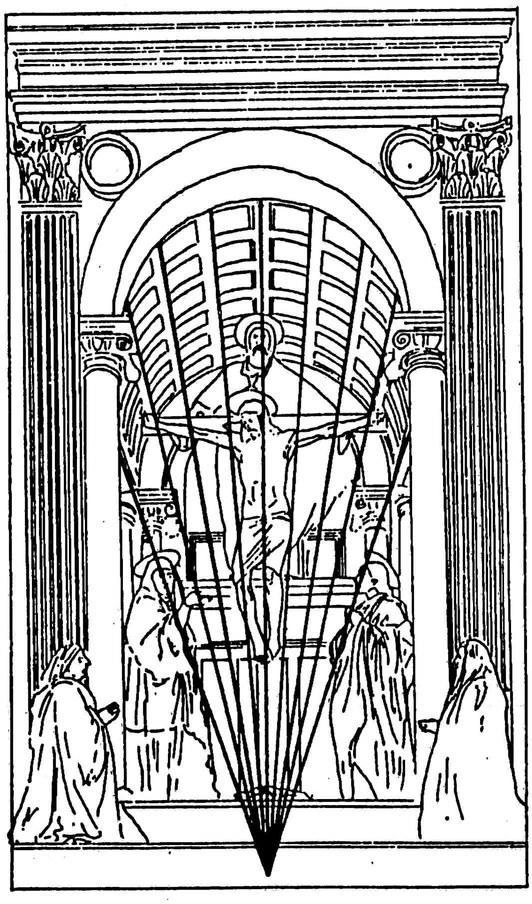
(ii) Hall of Kings (14th century), Alhambra, Spain.



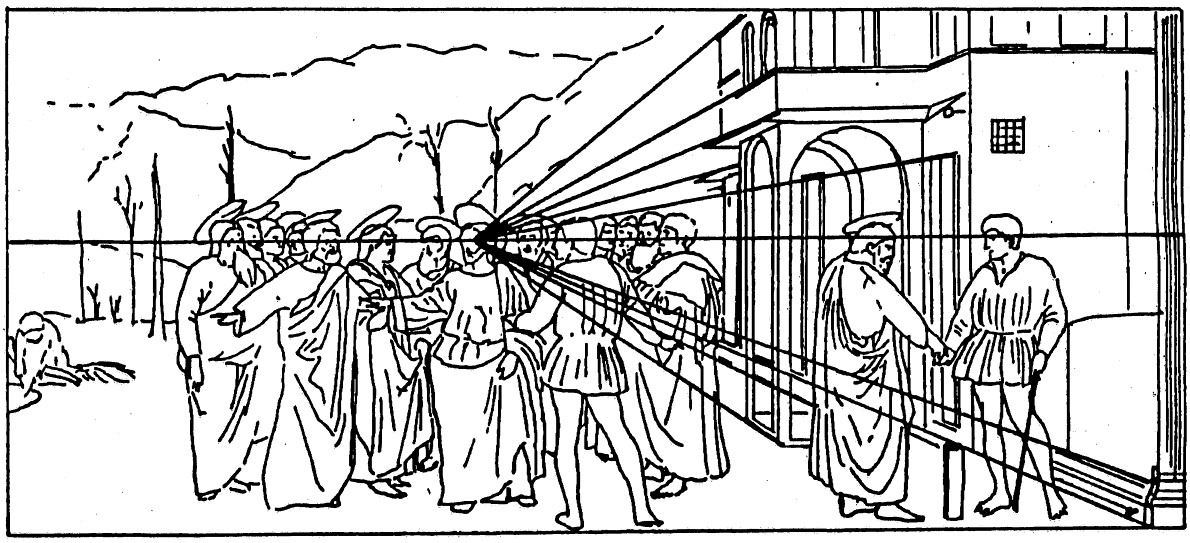


(iii) Roman wall-painting (Augustan period 27BC, uncovered 1961), Rome; illustrated in *Encyclopedia of World Art*, McGraw Hill (1966), Volume XI, Plate 87.

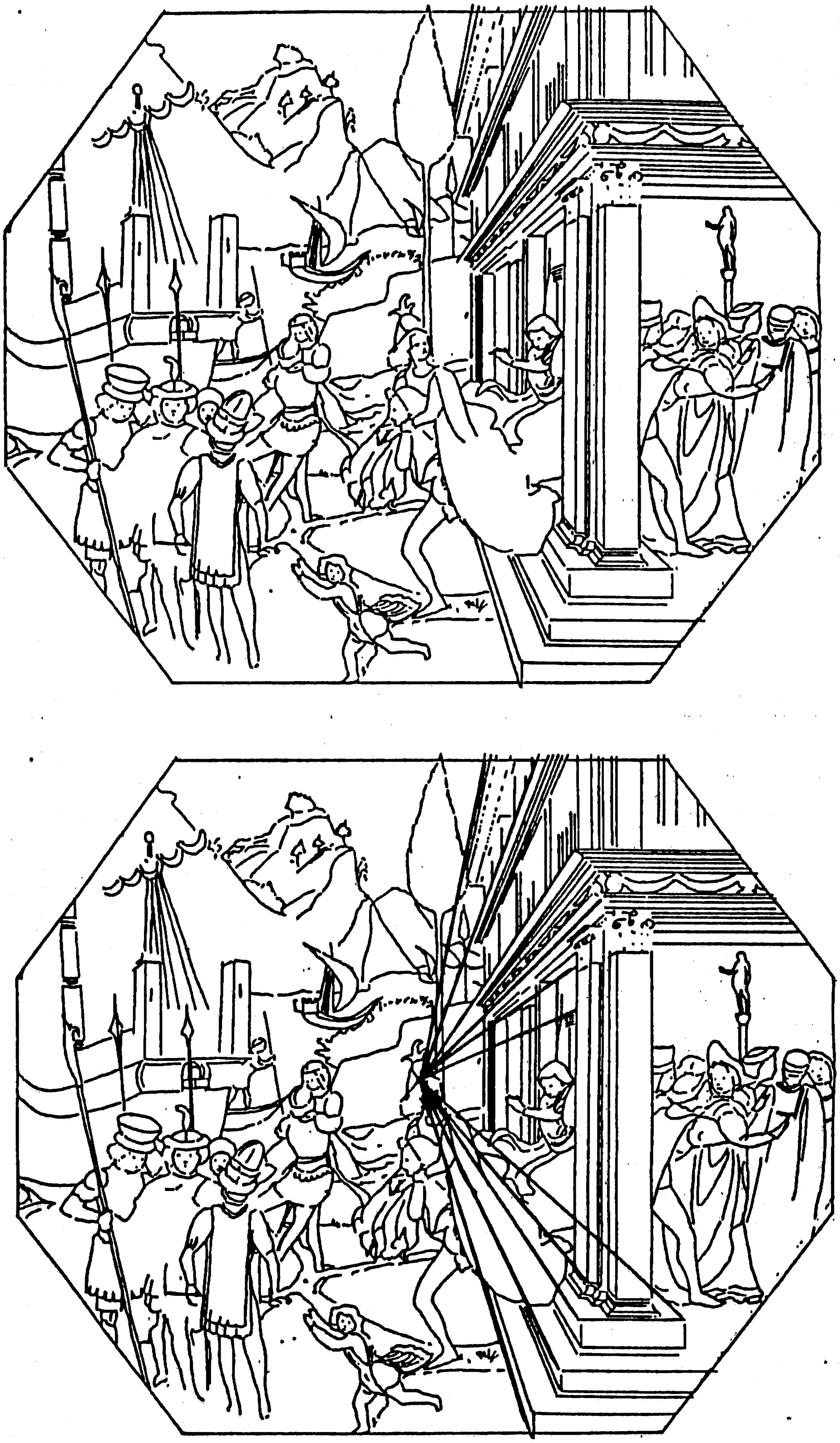




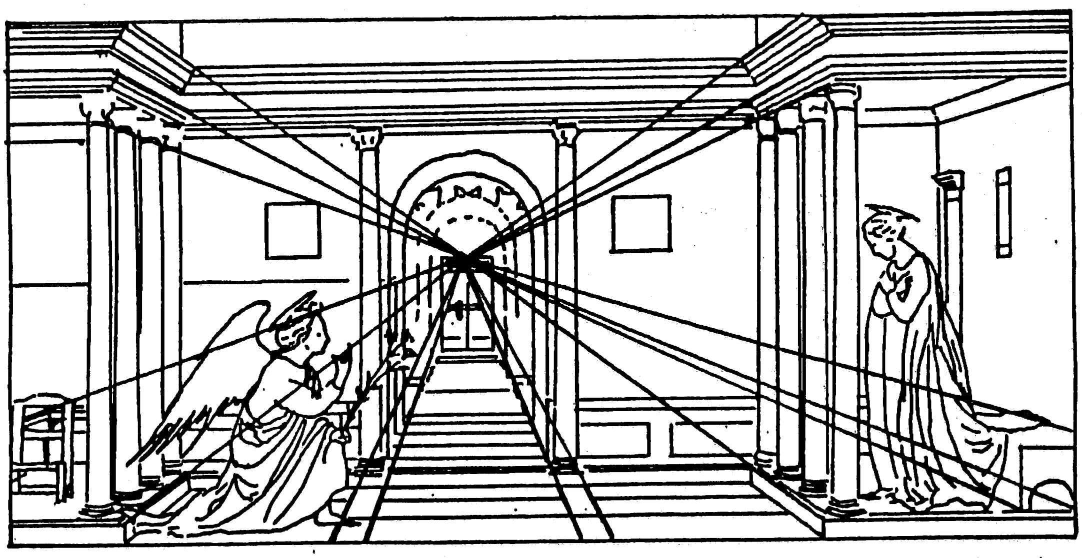
(iv) Masaccio (1401-1428), *Trinity* (1427).
Santa Maria Novella Church, Florence.



(v) Masaccio, *The tribute money* (1427), Brancacci Chapel, Florence.

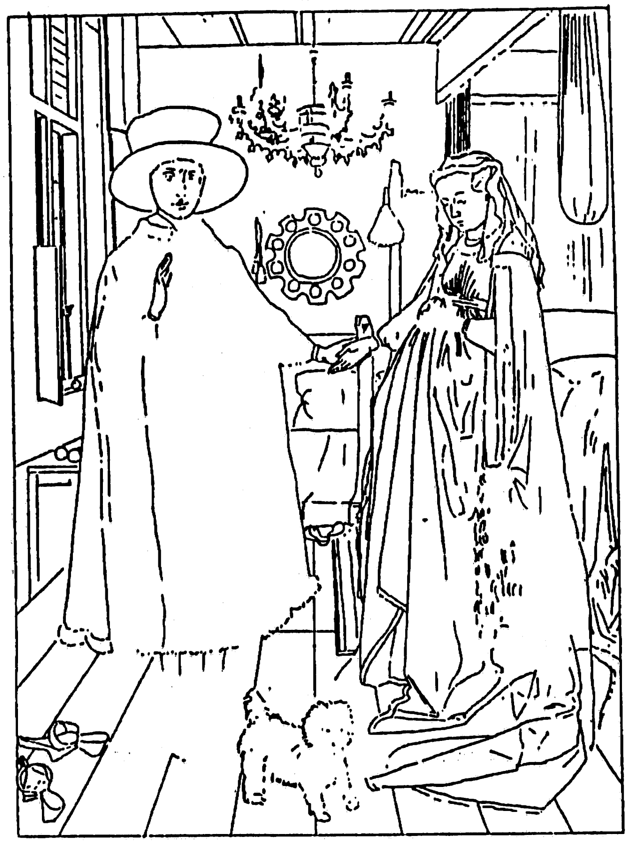


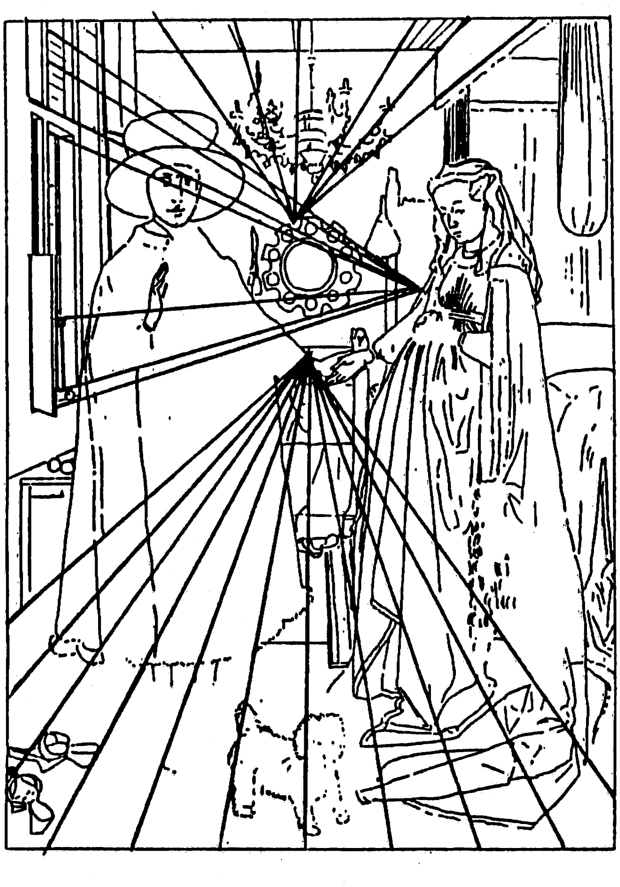
(vi) Gozzoli (1427-1497), ***The abduction of Helen by Paris***,
National Gallery, London (number 591).



(vii) Domenico Veneziano (active 1438-1461), *The Annunciation*, Fitzwilliam Museum, Cambridge (number 13).

(viii) Domenico Veneziano, *A miracle of St. Zenobius*, Fitzwilliam Museum, Cambridge (number 101)





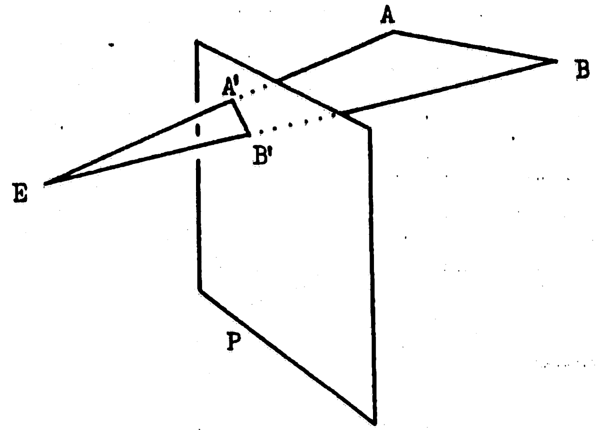
(ix) Van Eyck (active 1422-1441), *Marriage of Arnolfini* (1434), National Gallery, London (number 186).

Geometry and perspective

Mathematical Notes 1.

Let P be the plane of the picture, which it is useful to think of as a pane of glass, and let E be the position of the eye.

Definition of image.

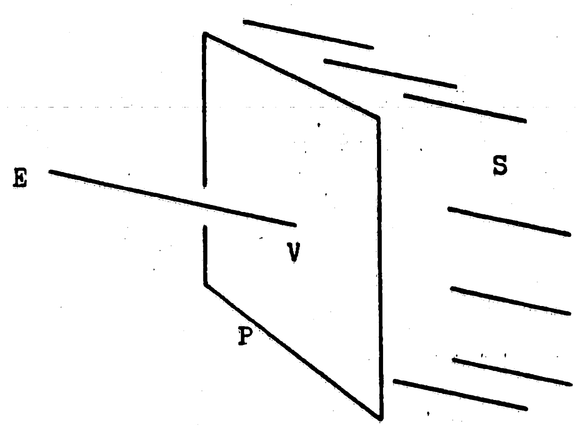


Define the image of a point A to be the point A' where the ray EA pierces P. Similarly the image of B is the point B' where EB pierces P.
Define the image of the line AB to be the line A'B'.

Remark. Mathematically the word "ray" means the same as the word 'line', and we could just as well have written "the image of A is the point A' where the line EA meets P". However, I prefer the more colourful language of a ray piercing P because it evokes the idea of looking at A through the pane of glass. Of course in terms of physics the light goes from A to E, but in terms of psychology the focus of attention goes the other way from E to A, and ordinary language tends to reflect psychology rather than physics: we actively look at something rather than passively turn our eyes to receive light signals from that object.

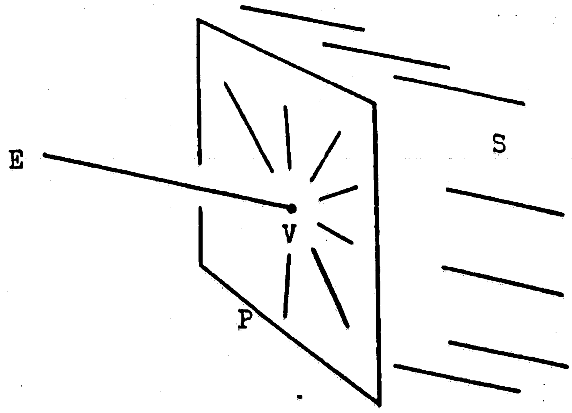
Definition of vanishing point.

Define the vanishing point of a set S of parallel lines to be the point V where the ray through E parallel to S pierces P.



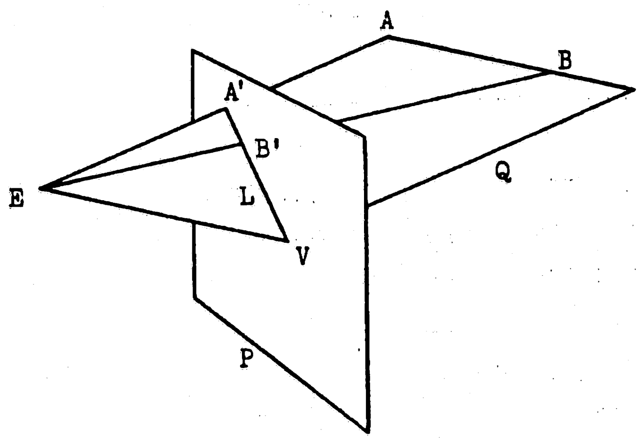
Theorem.

All the images of S, when extended, go through V.



Proof. It suffices to prove the theorem for just a single line, for if we prove that <u>one</u> image goes through V then by the same proof they <u>all</u> must.

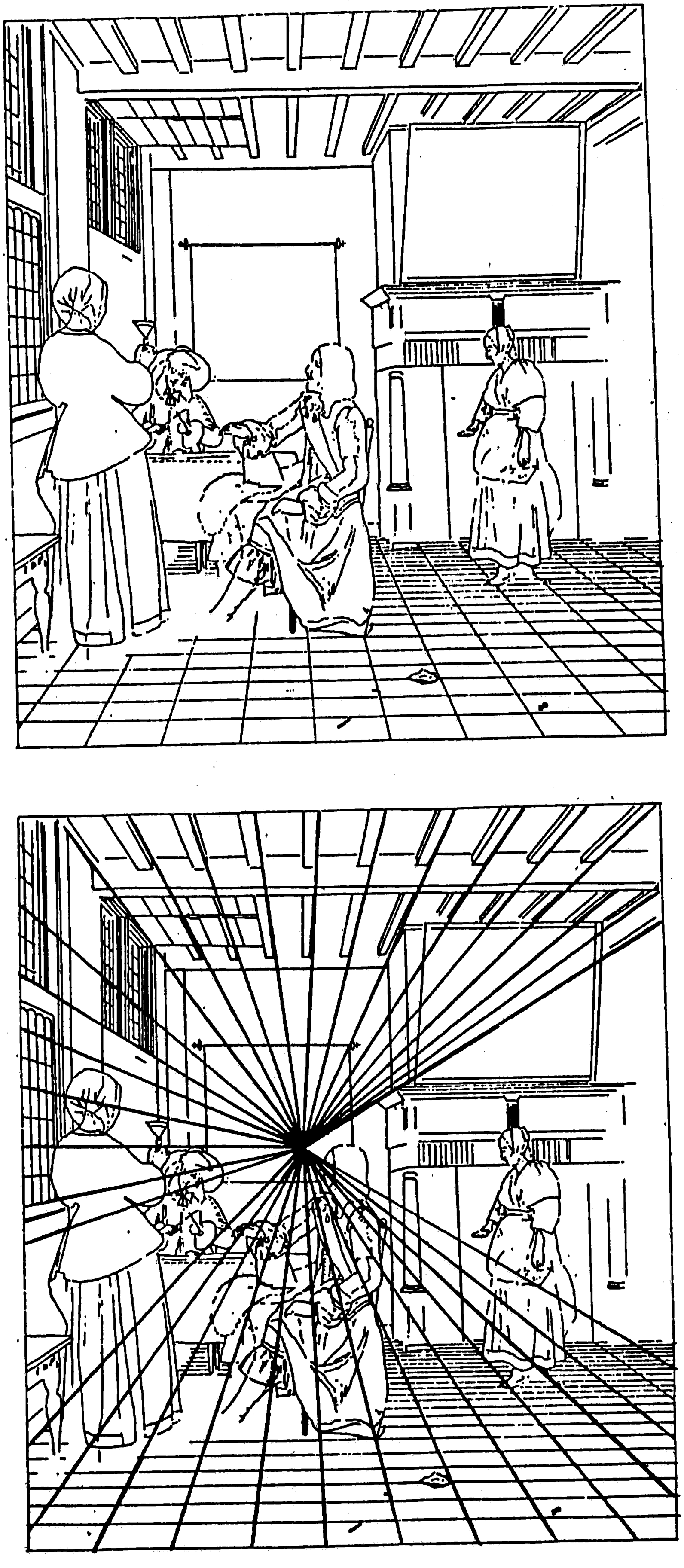
Let AB be one of the lines. The vanishing point V is where the parallel line through E pierces P. Any two parallel lines determine a plane. Therefore AB and EV determine a plane, which we call Q. Any two planes intersect in a line. Therefore P and Q intersect in a line, which we call L.



V lies in both P and Q, and therefore on L. A' lies in P and also on EA, which is contained in Q. Hence A' lies in both P and Q, and therefore on L. Similarly B' lies on L. Therefore A'B' lies along L. Therefore when A'B' is extended along L it goes through V. This completes the proof of the Theorem.

Example.

In the painting by de Hoogh sketched on the next page all the lines of the floor tiles, the window panes and the roof timbers are in reality parallel to each other. Therefore their images in the painting all go through a single vanishing point.



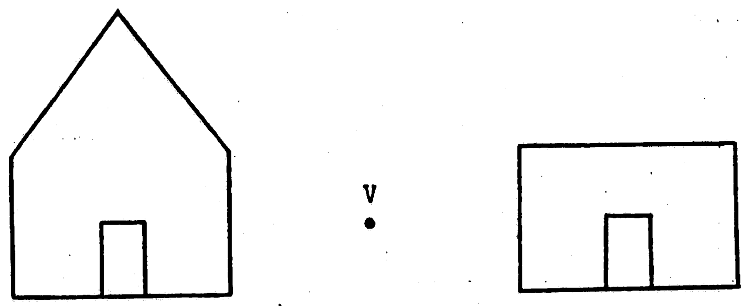
(1) Pieter de Hoogh (1629-1683): *An interior scene* (1655), National Gallery, London (number 834).

Geometry and perspective

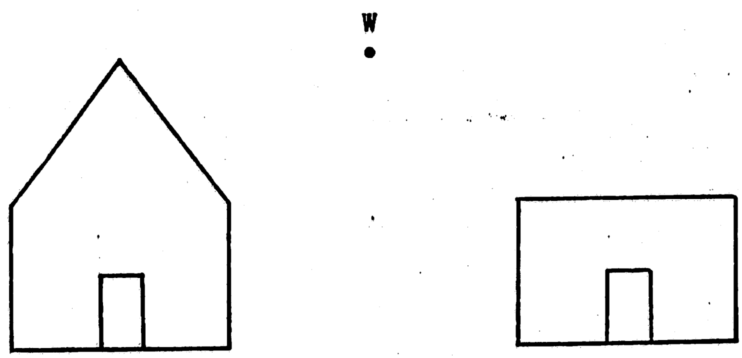
Worksheet 1.

Try and do all these questions after you have seen the video Part 1, and before you watch Part 2. The first question is a drawing exercise using a vanishing point, as you have already seen in Part 1. The rest of the questions introduce some mathematical ideas that you will be meeting in Part 2, and will give you a chance to think about them beforehand; this will enable you to understand Part 2 much more easily.

1. The two drawings below are exactly the same and show the fronts of two buildings, one with a pitch roof and the other with a flat roof. In the top drawing use the vanishing point V to draw the sides of the buildings facing each other, so that you get an eye-level perspective view of them.



In the bottom drawing use the vanishing point W to draw the sides, so that you now get an aerial perspective view of them.

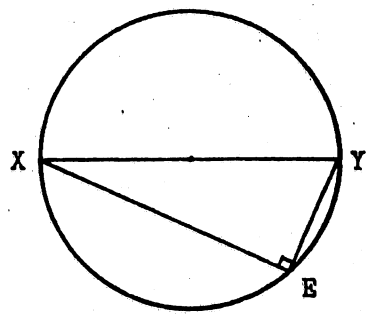


2. Draw an acute-angled triangle XYE, in other words one in which all the angles are smaller than a right-angle. Draw the circle with diameter XY and verify that E lies outside the circle.

Next draw an obtuse-angled triangle XYE with the obtuse angle at E; in other words E is greater than a right-angle. Draw the circle with diameter XY and verify that E now lies inside the circle.

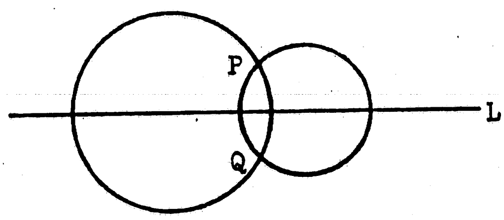
Finally draw a right-angled triangle XYE with the right-angle at E. Draw the circle with diameter XY and verify that this time E lies on the circle.

3. Prove that if EX,EY are at right-angles then E lies on the circle with diameter XY. [Hint : complete the rectangle.]



4. Deduce that if EX, EY are at right-angles then E lies on the sphere with diameter XY.

5. Suppose that two circles meet in the points P and Q. Let L denote the line through the centres of the circles. Prove that P is the reflection of Q in L.



6. If you spin the diagram in the last question around the axis L what do you obtain? Deduce that if two spheres meet and do not touch then they intersect in a circle.

7. What is the intersection of three spheres?

Geometry and perspective

Video Part 2 : Cubes and observation points.

In the first part of the video we looked at just one vanishing point in each picture. But the theorem says that any set of parallel lines has a vanishing point. Therefore a picture may have several vanishing points corresponding to several sets of parallel lines.

Now the simplest solid object with more than one set of parallel lines is a cube. A cube has altogether 12 edges, and each edge belongs to a set of 4 parallel edges. Therefore there are 3 sets of 4 parallel edges. Therefore to draw a cube you need 3 vanishing points, and in the video you will see how to do it. But that is not the end of the matter, because once you have drawn a cube in perspective you find that you then have to put your eye in a special position in order to see it in perspective. This special position is called the observation point. And to study the observation points we do some scientific experiments and prove a mathematical theorem.

In Worksheet 2 you will be invited to do some visual experiments to see for yourself the astonishing difference that it makes when you put your eye in the right position, and the heightened perception of 3-dimensionality that you then obtain. Meanwhile in the video and in Mathematical Notes 2 we prove the corresponding theorem, establishing the existence and uniqueness of the observation point.

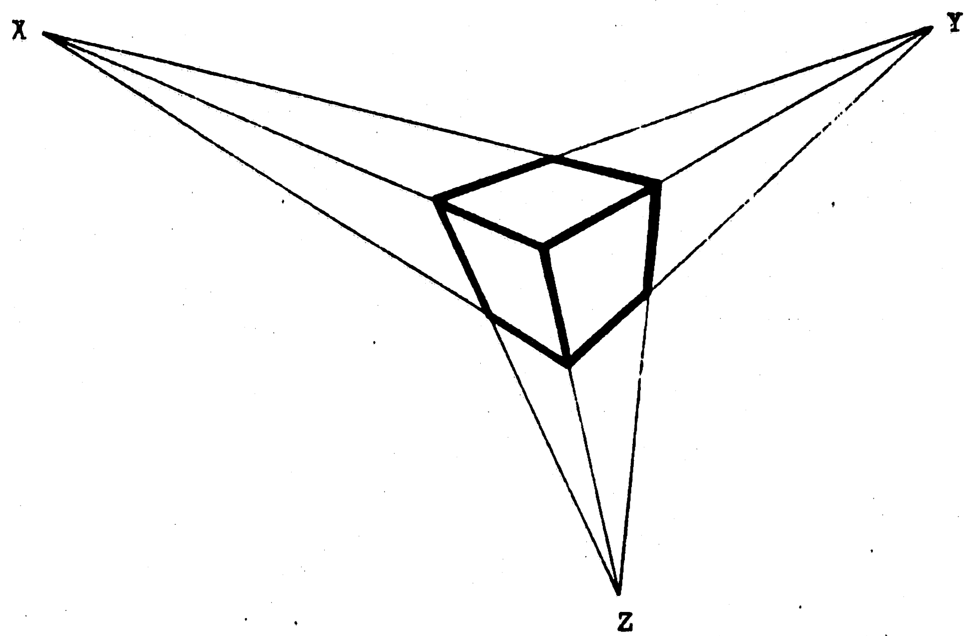
There is all the difference in the world between a scientific experiment and a mathematical theorem. Any scientific experiment must necessarily be confined to a particular observer observing a particular case at a particular place at a particular time, and must necessarily be subject to experimental error (as well as possible later refutation). A mathematical theorem, on the other hand, embraces all cases, and once proved remains true for ever. The theorem explains why the experiment works.

Geometry and perspective

Mathematical Notes 2.

Example 1.

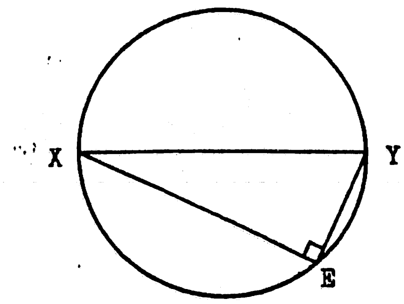
Let XYZ be an acute-angled triangle. Here is a drawing of a cube in perspective using X,Y,Z as vanishing points.



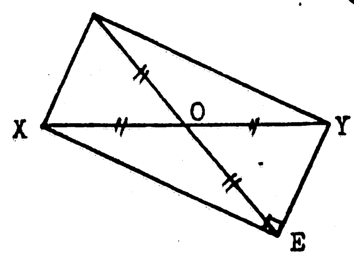
Let E be the position of the eye. By definition of vanishing point (in Mathematical Notes 1) EX is parallel to an edge of the cube. Similarly EY is parallel to another edge. But these two edges are at right-angles. Therefore EX,EY are at right-angles. Similarly with EZ. Therefore to see the cube in perspective you must place your eye at a point such that EX,EY,EZ are all at right-angles to each other. We call such a point an observation point.

Theorem 1. There is exactly one observation point.

Remark. To prove the theorem we shall first need to prove three lemmas. A "lemma" means a little subsidiary theorem needed for the proof of the main theorem.



Lemma 1. If EX, EY are at right-angles then E lies on the circle with diameter XY.



Proof. Complete the rectangle. By symmetry the diagonals are equal and bisect each other, at a point O say.
∴ OX = OY = OE.
Therefore the circle with centre O and radius OX passes through E. But this _is_ the circle with diameter XY.

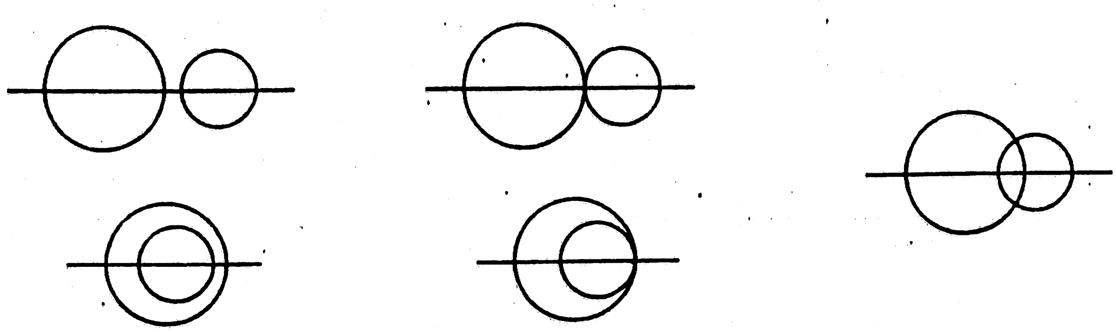
Lemma 2. If EX, EY are at right-angles then E lies on the _sphere_ with diameter XY.

Proof. Spin the circle in the last lemma about the axis XY.

Lemma 3. If two spheres meet they either touch at one point or intersect in a circle.

Proof. Let L be the line through the centres of the spheres, and let P be a plane containing L. Each sphere meets P in a circle with the same centre. If we spin these circles around L we recover the spheres. Therefore if we spin the intersection of the circles around L we recover the intersection of the spheres.

Now the circles either miss each other, or touch, or intersect in two points as shown.

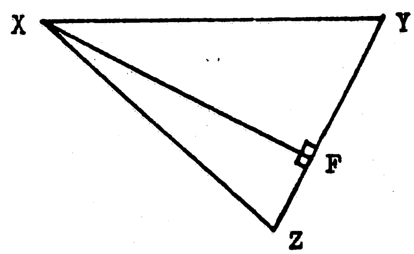


If they miss then so do the spheres. If they touch then so do the spheres. If they intersect then the two points are equidistant from L by symmetry, and so spin into the same circle. Therefore the two spheres intersect in this circle.

<u>**Proof of Theorem 1.**</u>

Suppose E is an observation point. Then EX, EY are at right-angles. Therefore, by Lemma 2, E lies on the sphere diameter XY. Similarly E lies on the spheres diameter XZ and YZ. Therefore E lies on the intersection of all three spheres. Hence we must look for the intersection of the spheres.

Now the first two spheres with diameters XY and XZ must meet because they both contain X. Let C denote their intersection. Then C is either a point or a circle by Lemma 3. We shall prove C must be a circle by showing that it contains another point besides X.



Let F be the foot of the perpendicular from X to YZ. In other words, XF is the <u>altitude</u> of the triangle XYZ through X. Since the angles Y and Z are both acute F must lie in between Y and Z. Since FX,FY are at right-angles F lies on the sphere diameter XY, by Lemma 2. Similarly F lies on the sphere diameter XZ. Therefore F lies on their intersection C. Therefore C is a circle.

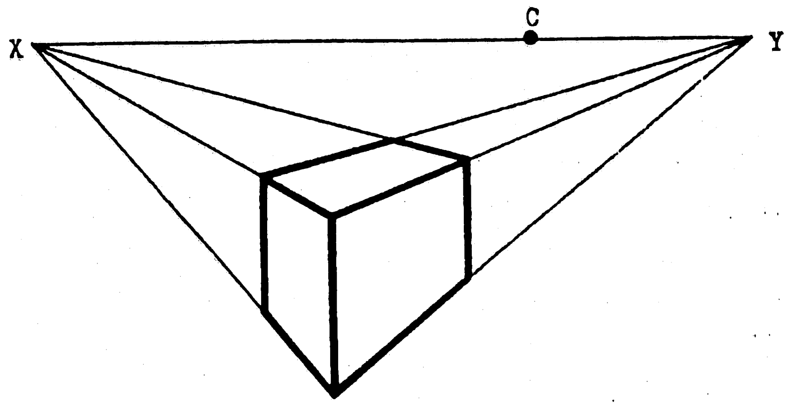
Let S be the third sphere with diameter YZ. Then the intersection of C and S will give the intersection of all three spheres. Now F lies inside S because it lies in between Y and Z. Meanwhile X lies outside S because the angle of the triangle XYZ at X is acute. Therefore C contains points both inside and outside S. Therefore C must pierce S in two points. Therefore all three spheres meet in these two points.

Since the whole configuration is symmetrical with respect to the plane P containing XYZ, one of these points must lie in front of P and the other one must be its mirror image behind P. But we are only interested in points in front of P, because we want to observe the drawing on the front of P. Therefore there is only one possible position for the observation point E. This completes the proof of Theorem 1.

__Remark.__ If XYZ is an obtuse-angled triangle there is no observation point because the three spheres do not meet (see Worksheet 2 question 9).

__Example 2.__

Here is a drawing of a cube in perspective using X,Y as vanishing points, and with the other edges drawn perpendicular to XY.



__Theorem 2.__ The figure looks rectangular when viewed from any point on the semi-circle with diameter XY perpendicular to the paper.

__Proof.__ For convenience of language assume the paper is vertical with the line XY horizontal. We can deduce this result as a limiting case of the previous result as Z descends to minus infinity. As Z descends the sphere with diameter XZ grows, and tends to the horizontal plane through X. Similarly the sphere with diameter YZ tends to the horizontal plane through Y. These two planes are the same because XY is horizontal. The third sphere with diameter XY meets this plane in the horizontal circle with diameter XY. Taking that part of the circle in front of the paper gives the theorem.

__Remark.__ As the eye moves round the viewing semi-circle the apparent dimensions of the box change: at the left end it looks more like a matchbox, and at the right end it looks more like a cube. More precisely, there is only one observation point from which it looks exactly cubical, and that is the point in front of C on the semi-circle.

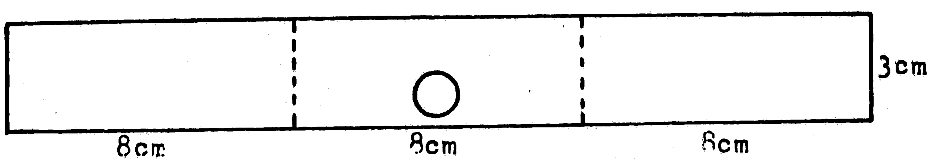
Geometry and perspective

Worksheet 2.

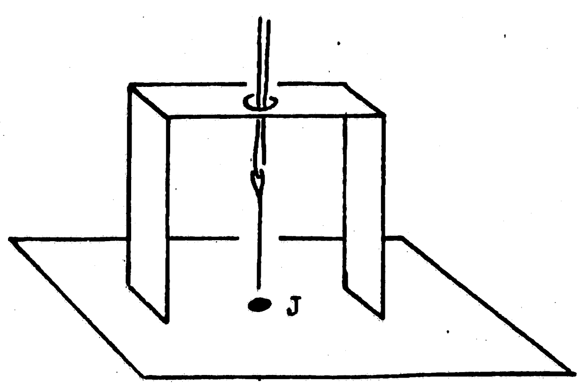
This Worksheet is mostly about drawing cubes in perspective, as you saw in the Video Part 2. Questions 1-4 are about visual experiments; Questions 5-10 involve mathematical proofs about the observation point; and Questions 11-17 are about the numerical calculations necessary to get the lengths of the edges of the cube in the drawing correct, so that when you view it from the observation point it will really look cubical rather than merely rectangular. Question 18 is about making some axes to do the experiment of drawing a cube on a blackboard, as in the video. Finally Question 19 explains how to make a perspective drawing of the Baptistry in Florence.

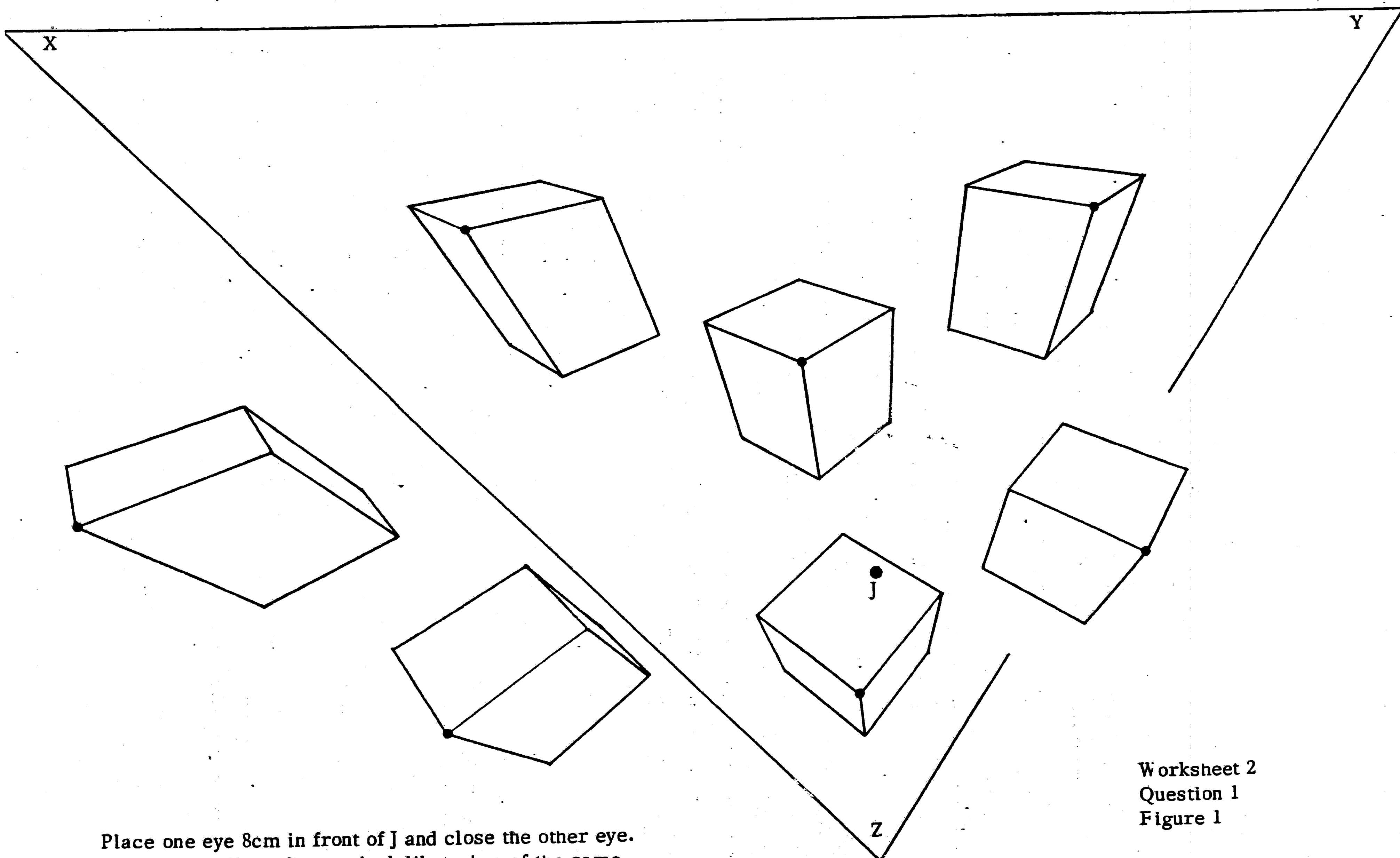
It is not strictly necessary to do all the questions in this worksheet before seeing the Video Part 3. However, it is very important to get used to the idea of using more than one vanishing point, and the idea that you have to put your eye in exactly the right spot before you can see a picture properly in perspective. The more questions you try the more familiar you will become with these ideas, and the easier you will find the last part of the video. And you will especially enjoy Brunelleschi's experiment if you first do Question 19.

1. Make a little peephole 8cm high as follows. Cut a strip of cardboard 24cm × 3cm and divide it into three with score-lines: by a "score-line" I mean cut half-way through the cardboard with a sharp knife.



Make a peephole in the middle with a hole puncher. Then bend along the score lines and stand it over the point J in Figure 1 on the next page. You can make sure that it is exactly





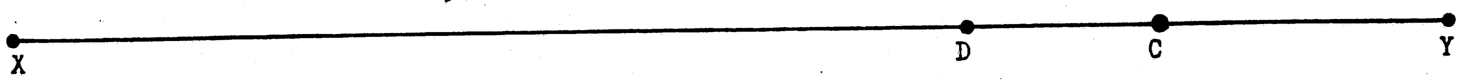
Worksheet 2
Question 1
Figure 1

Place one eye 8cm in front of J and close the other eye. Verify that all the figures look like cubes of the same size with parallel faces.

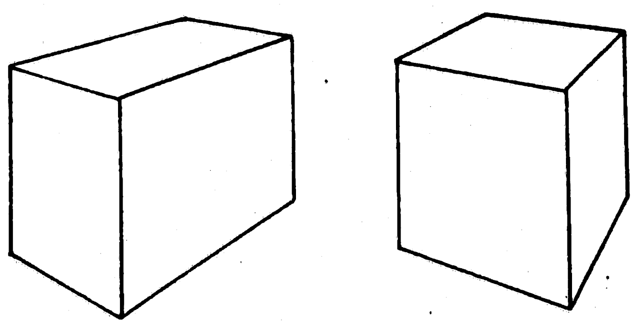
over J with the help of a plumb line made out of needle and cotton. Place one eye at the peephole and close the other eye, and looking through the peephole verify that each of the figures looks like a cube. You may have to rotate the peephole to see the outermost figures.

Then, without moving your eye, gently remove the peephole and verify that all the figures simultaneously look like cubes of the same size with all their faces parallel. You can move your eye around a little to make sure you have found exactly the right spot where they all look cubical. Finally move your eye gently away and watch them becoming distorted as they move out of perspective.

2. Use your 8cm peephole above the point C in Figure 2 below, and verify that each of the figures looks like a cube. Verify that both cubes look the same size with their faces parallel. The difference between this question and the last is that here the vertical edges have been drawn parallel: that is why the peephole has to be put over a point on the line XY.

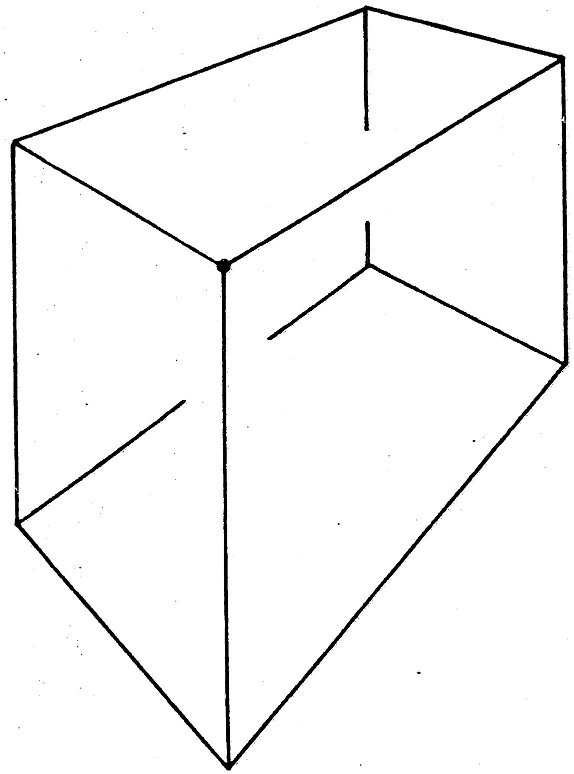


Worksheet 2
Question 2
Figure 2



3. Measure XY in Figure 3 on the next page, and make a semi-circle of wire with XY as diameter. Hold the semi-circle perpendicular to the paper, place one eye next to the wire and, closing the other eye, verify that the figure looks like a rectangular box. Now slide your eye along the wire and verify that although the box always looks rectangular its dimensions appear to vary continuously.

A simple way to do this experiment (without the wire) is to hold the paper vertical with one eye on the (imaginary) semi-circle; then, keeping the eye still, rotate the paper about the vertical axis through its mid-point so that, relative to the moving paper, the fixed eye



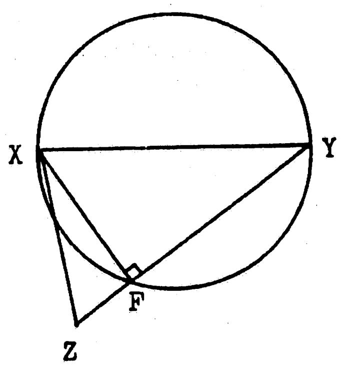
Place one eye 9.6cm in front of C and close the other eye. Verify that the figure looks cubical. Rotate the paper so that eye slides round the semi-circle diameter XY, and verify that the box continues to look rectangular but its dimensions appear to vary.

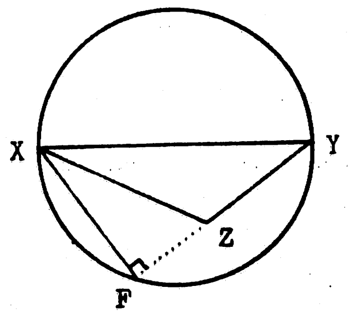
Worksheet 2
Question 3
Figure 3

slides along the imaginary semi-circle. Verify that when the eye is near X the figure looks like a matchbox and near Y it looks like a cube. More precisely, verify that it only looks exactly cubical from one point, namely the point opposite C on the semi-circle. If you want to check it with a peephole then the peephole should be 9.6cm high and placed exactly over C.

4. In Figures 2 and 3 above measure XC and CY and calculate $h = \sqrt{XC.CY}$. Verify that this formula gives the height h of the peephole.

5. Let XF be the altitude of the triangle XYZ through X. Prove that F lies on the circle with diameter XY. Verify that your proof is valid for both acute-angled and obtuse-angled triangles.



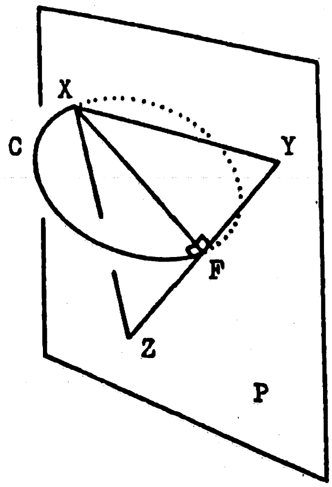


6. Draw an acute-angled triangle XYZ. Draw the circles with diameter YZ,ZX,XY, and verify that they meet in pairs at points F,G,H on the side YZ,ZX,XY, respectively. Draw XF,YG,ZH and verify that they all meet at a point J.

Deduce from the last question that XF,YG,ZH are the altitudes of the triangle. The point J is called the <u>orthocentre</u> of the triangle.

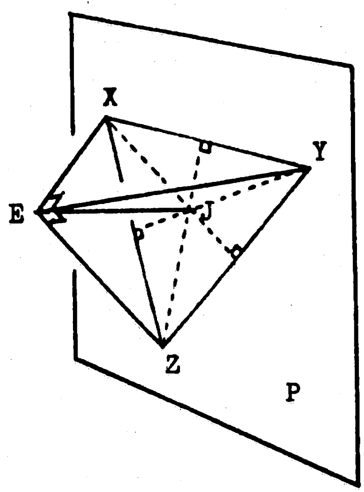
Use this construction to verify that the point J in Figure 1 of Question 1 is the orthocentre of XYZ.

7. Let XF be the altitude of the triangle XYZ through X. Let P be the plane containing XYZ. Let C be the circle of intersection of the spheres diameters XY and XZ. Prove that C is the circle with diameter XF perpendicular to P. [Hint: use reflection in P.]

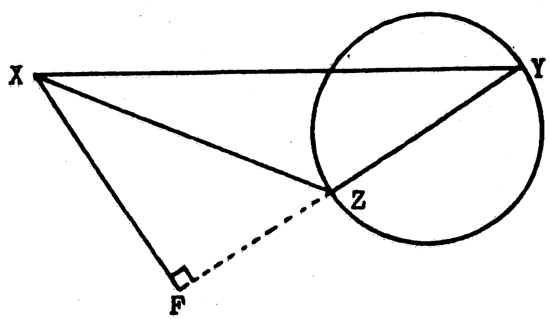


8. Suppose that XYZ is an acute-angled triangle. Deduce from the last question that the foot of the perpendicular from the observation point E to the plane P is the orthocentre J of the triangle XYZ.

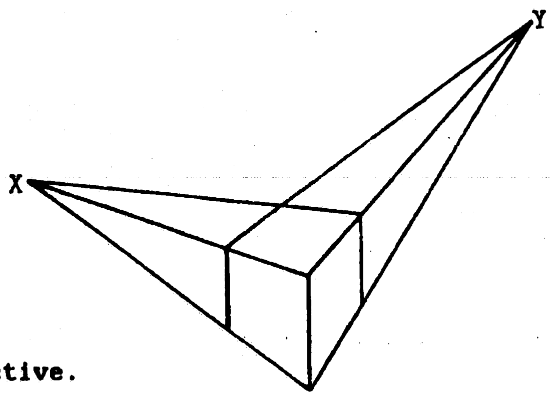
This explains why the peephole in Question 1 had to be placed over the orthocentre J.



9. Prove that if XYZ has an obtuse angle at Z then the altitude XF lies outside the circle diameter YZ. Deduce that there is no observation point in this case.

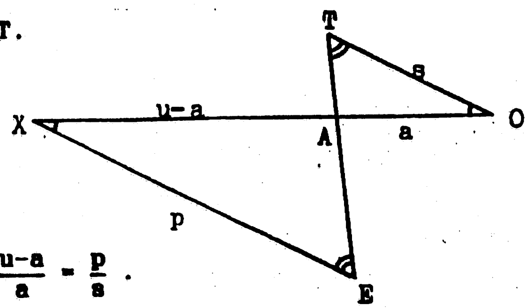


10. Prove that if a cube is drawn using X,Y as vanishing points, and with the other edges drawn parallel but <u>not</u> perpendicular to XY then there is no observation point from which it can be seen in perspective. How is this related to the previous question?



11. In the diagram EX is parallel to OT.

Let OA = a
OX = u
EX = p
OT = s.



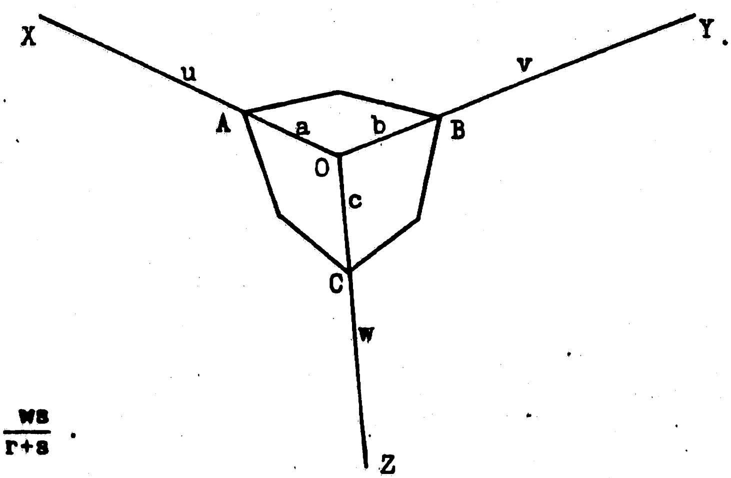
Prove by similar triangles that $\frac{u-a}{a} = \frac{p}{s}$.

12. Deduce from the last question that $\frac{u}{a} = \frac{p+s}{s}$, and hence $a = \frac{us}{p+s}$.

13. Suppose that a cube has one vertex O in contact with a pane of glass P, and is observed from a point E on the other side of the glass. Let OA,OB,OC be the edges of the resulting perspective drawing of the cube on P, and let X,Y,Z be the corresponding vanishing points, as shown.

Let s = side of cube, and

OA = a	OX = u	EX = p
OB = b	OY = v	EY = q
OC = c	OZ = w	EZ = r



Deduce from the last two questions that :

$a = \frac{us}{p+s}$, $b = \frac{vs}{q+s}$, $c = \frac{ws}{r+s}$.

14. Using the notation of the last question for Figure 1 in Question 1, the distances in centimetres of the observation point E from the vanishing points X,Y,Z are :

$$p = 21.9 \quad , \quad q = 17.4 \quad , \quad r = 9.8 \quad .$$

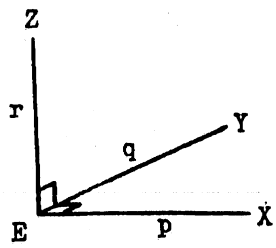
All the cubes have side s = 3. For each cube measure u,v,w and use the formulae in the last question to calculate a,b,c to the nearest millimetre. Verify your answers by measuring a,b,c.
[Warning: in each case O has to be the vertex such that the edges OA,OB,OC point towards X,Y,Z, and for convenience this vertex has been marked with a dot.]

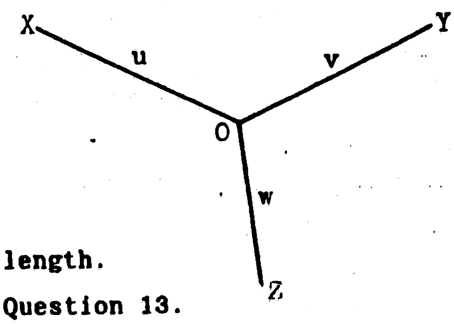
15. In Figure 2 of Question 2 the vertical edge OC of each cube is in contact with the pane of glass, and so c = s = 3. The distances of the observation point E from the vanishing points X,Y are approximately p=18, q=9. For each cube measure u,v and use the formulae in Question 13 to calculate a,b. Verify your answer by measuring a,b.

16. Do the same in Figure 3 of Question 3 given that this case s = 7, p = 21.5, q = 10.7.

17. Prove that if $\frac{r}{w} \to 1$ as $w \to \infty$ then $\frac{ws}{r+s} \to s$.
Given a geometrical interpretation of this in terms of Question 13 and Figure 3.

18. Make some 3-dimensional rectangular axes. Measure the lengths p,q,r of the axes. Place the ends of the axes on a blackboard, or a large piece of paper, and mark the points where the ends come as X,Y,Z. Choose a vertex O inside the triangle XYZ and measure u,v,w.





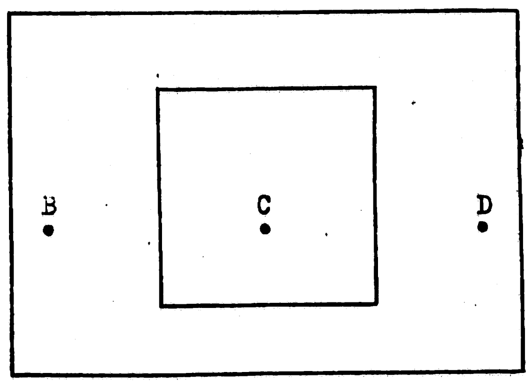
Choose s = 15cm, or any convenient length. Calculate a,b,c by the formulae in Question 13. Mark the points A,B,C on the lines OX,OY,OZ distances a,b,c from O. Complete the perspective drawing of the cube as in Question 13, using X,Y,Z as vanishing points. Replace the axes and look at the the drawing with one eye placed as near to E as possible. Verify that it looks cubical. Remove the axes and move your eye around gently to see how the cube moves in and out of perspective.

If the paper is horizontal hang a plumb line from E and mark the end J. Verify that J is the orthocentre of XYZ, as in Question 6.

19. Take a piece of paper more than 24cm long and turn it sideways.

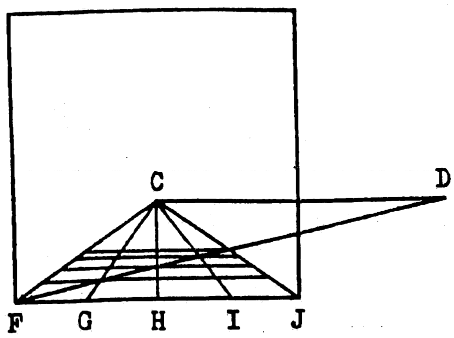
Draw a square in the middle of side 12cm.

Mark a point C in the centre 4cm above the bottom line of the square.



Mark points B and D at the same level and 12cm on either side of C.

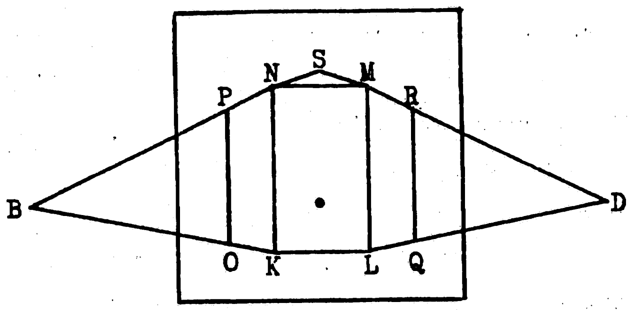
Divide the bottom of the square into quarters with points FGHIJ and join them to C.



Join DF.

Draw very carefully the horizontal lines through the 4 points where DF meets CG, CH, CI, CJ.

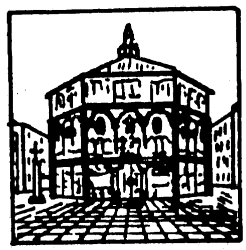
With its base resting on the topmost of these lines draw a rectangle KLMN of width 4cm and height 7cm.



Join BK, BN, DL, DM.

On BK,DL mark points O,Q such that OK = QL = 2cm, and draw verticals OP,QR as shown.

Choose a point S just above MN, and just below the intersection of BN and DM. Join S to M, N.



Add a little artistry to make a drawing of the Baptistry in Florence.

Make a peephole 12cm high and place it over C. Verify that the drawing looks in perspective through the peephole.

Geometry and perspective

Video Part 3 : Viewing distances and Brunelleschi's experiment.

The question arises: do the paintings we were looking at in Part 1 of the video also have a unique observation point? The answer is yes: each painting has its own unique viewing distance and observation point, and if you want to see it properly in perspective that is where you have to put your eye.

The easiest way to see this is to think about diagonal vanishing points. For example if you want to paint a room with a square tiled floor then the diagonals of the floor tiles are all parallel, and so when you paint them they should all go through a vanishing point. In the Video Part 3 and in Mathematical Notes 3 we prove a simple theorem relating such diagonal vanishing points to the viewing distance. From this you can easily work out the viewing distances of various famous paintings in museums and art galleries. It also explains the method of Alberti (1404-1472) for drawing a tiled floor, which he published in 1434 probably after learning it from Brunelleschi, and which is described in the video and Mathematical Notes 3.

We are now at last in a position to appreciate the genius of Brunelleschi, not only for the importance of his discovery but also for his ingenuity in solving the problem of how to convince his sceptical fellow artists. We can understand why he chose to paint the Baptistry in Florence. In the video you will see a reconstruction of his experiment, and Mathematical Notes 3 explains in detail how it worked.

If you want to read more about perspective I recommend *The Oxford Companion to Art* (Oxford University Press, 1970) pages 840-861, and if you want to read more about geometry I recommend D.Hilbert and S.Cohn-Vossen *Geometry and the imagination* (Chelsea, New York, 1952).

I hope you have enjoyed watching the video as much as I enjoyed making it. The accompanying music was Bach's Third Brandenburg Concerto, which he wrote in 1720, exactly 300 years after Brunelleschi's discovery.

Geometry and perspective.

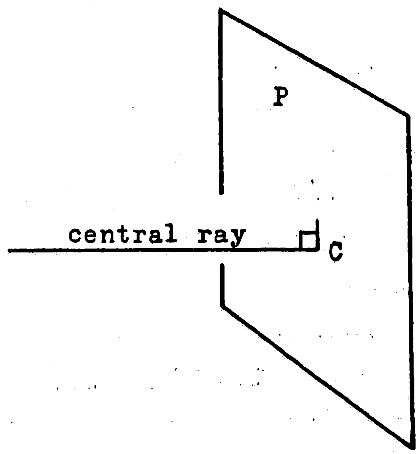
Mathematical Notes 3.

Definition.

The viewing distance is the distance from the eye to the picture.

Definition.

Let P be the plane of the picture, and let C be the vanishing point of the set of parallel lines perpendicular to P. C is called the central point, and the line through C perpendicular to P is called the central ray.



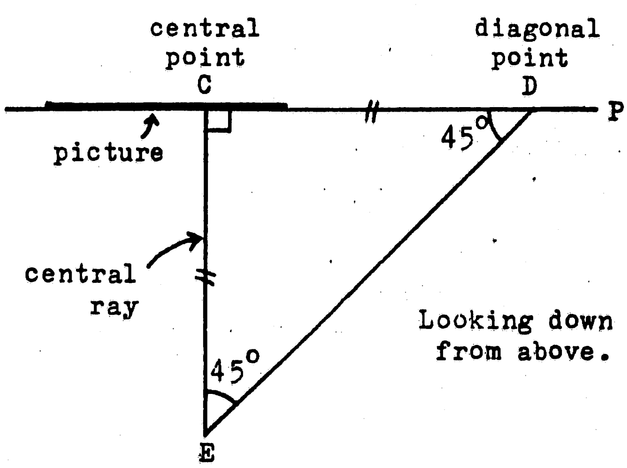
Definition.

Let D be the vanishing point of a set of parallel lines at 45^o to the central ray. D is called a diagonal point. Note that although D lies in the plane P of the picture it may in fact lie outside the picture itself.

Theorem. If C and D are given then there is exactly one observation point E, which lies on the central ray at viewing distance equal to CD.

Proof.

By definition of vanishing point in Mathematical Notes 1 EC is parallel to the set of lines perpendicular to P. Therefore E lies on the central ray.



By the definition of vanishing point D, ED must be inclined to 45^o to be central ray.

$\therefore$ $C\hat{E}D = 45^o$

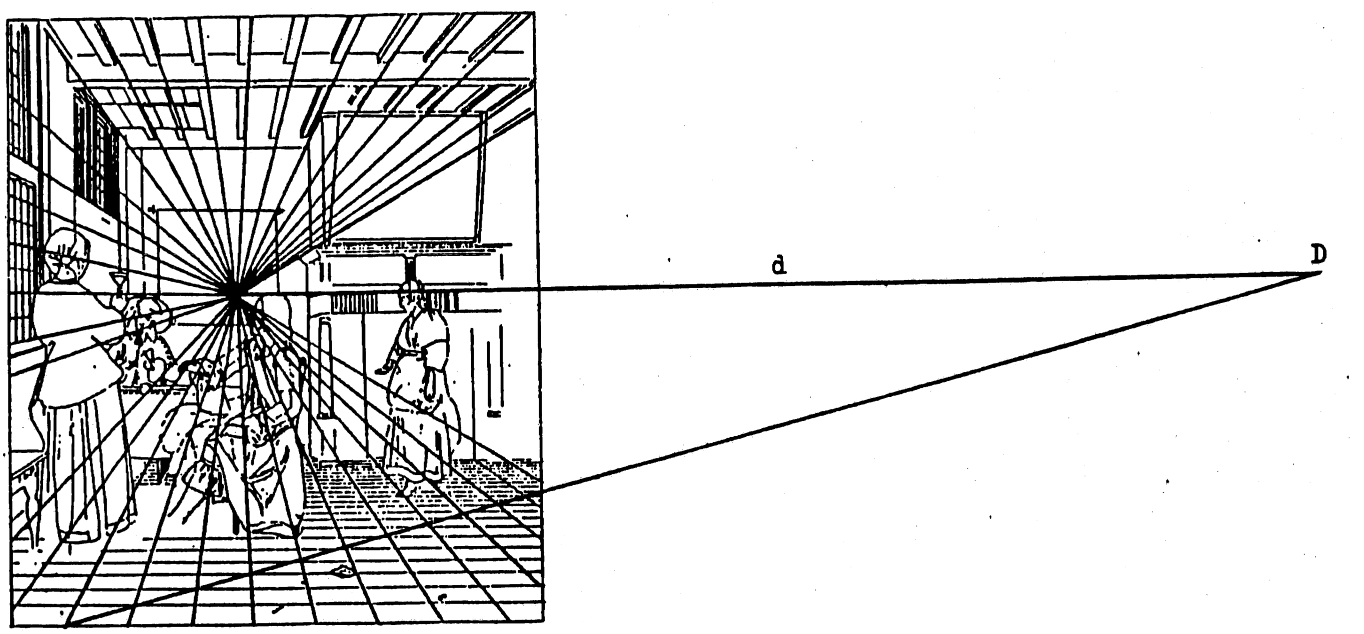
But $E\hat{C}D = 90^o$

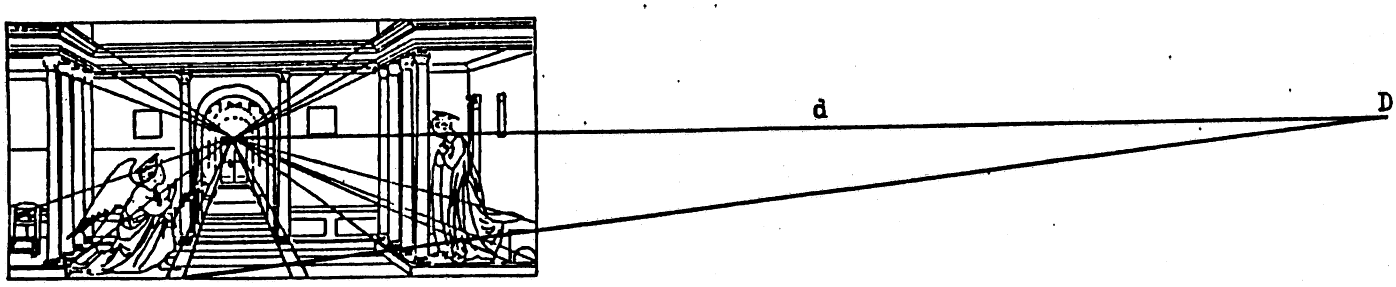
$\therefore$ $C\hat{D}E = 45^o$

$\therefore$ CDE is an isosceles triangle.

$\therefore$ EC = CD, as required.

__Examples.__ The sketches of de Hoogh's *Interior*, and Veneziano's *Annunciation* below show examples of central and diagonal points. The diagonal point D in each case is the intersection of the horizontal line through the central point C with the diagonal of one of the floor tiles, suitably extended. There is symmetrical diagonal point to the left, corresponding to the opposite diagonals.





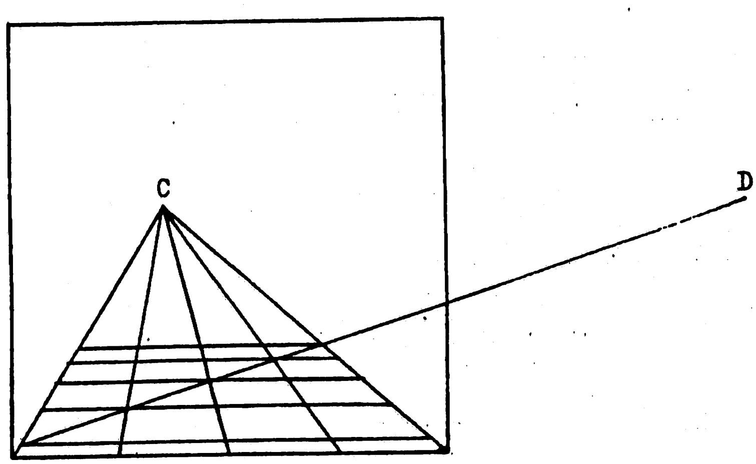
Application 1. How to find the viewing distance of a painting in a museum.

Buy a postcard reproduction of the painting. Draw in lines to find the central point C. Stick or clip the postcard onto a sufficiently large piece of paper, and draw in diagonal lines to find a diagonal point D on the same level as C. If D lies off the postcard it will be necessary to extend the lines onto the paper in order to find D. Measure d = CD, and this will give the correct viewing distance from which to see the postcard in perspective (with one eye in front of C and the other eye closed).

You can then find the viewing distance of the original painting by scaling up. More precisely measure the width w of the reproduction, and ask the museum curator for the width W of the original. Then the viewing distance of the original will be (W/w)d.

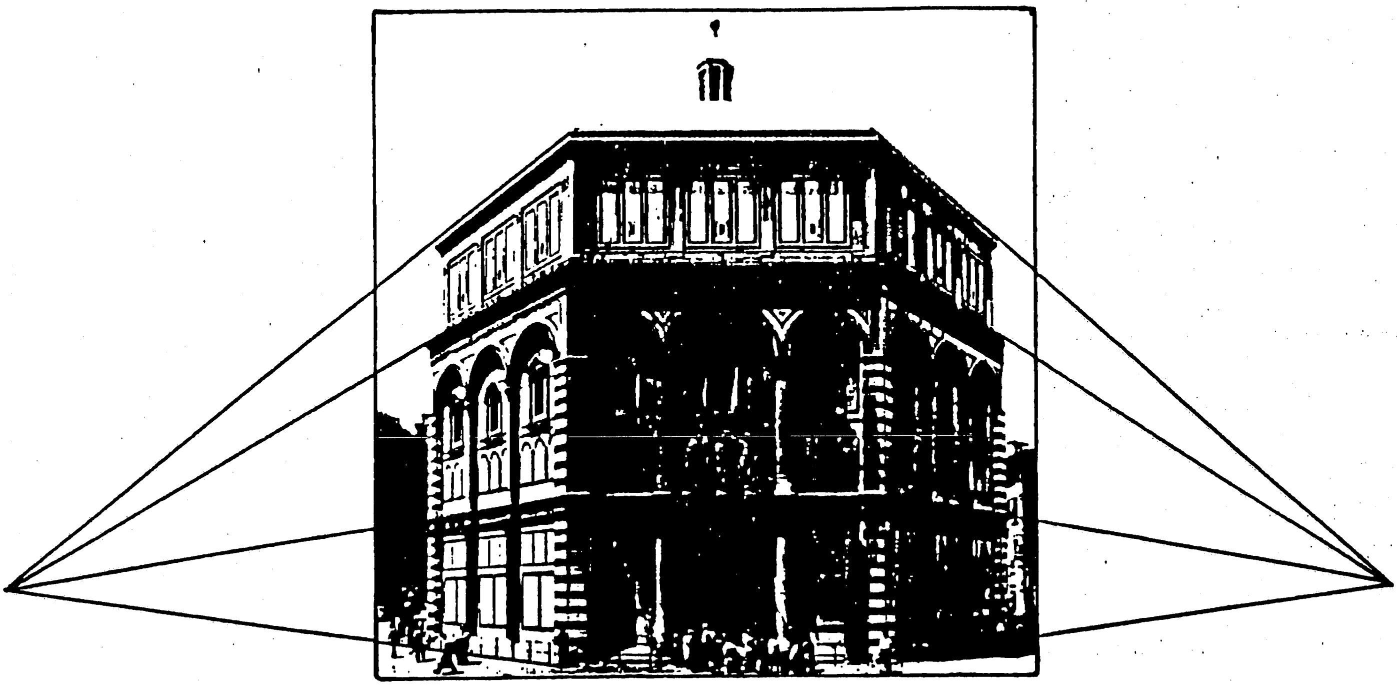
Application 2. How to draw a tiled floor.

Choose a central point C. Divide the bottom of the picture into equal parts, and join to C. Choose a diagonal point D level with C, and choose any line through D. Draw horizontals through the point where this line intersects the lines through C. Place your eye at a viewing distance equal to CD in front of C, and you will see a horizontal square-tiled floor in perspective. In essence this was the method described by Alberti in 1436 in his book *Della Pittura*.



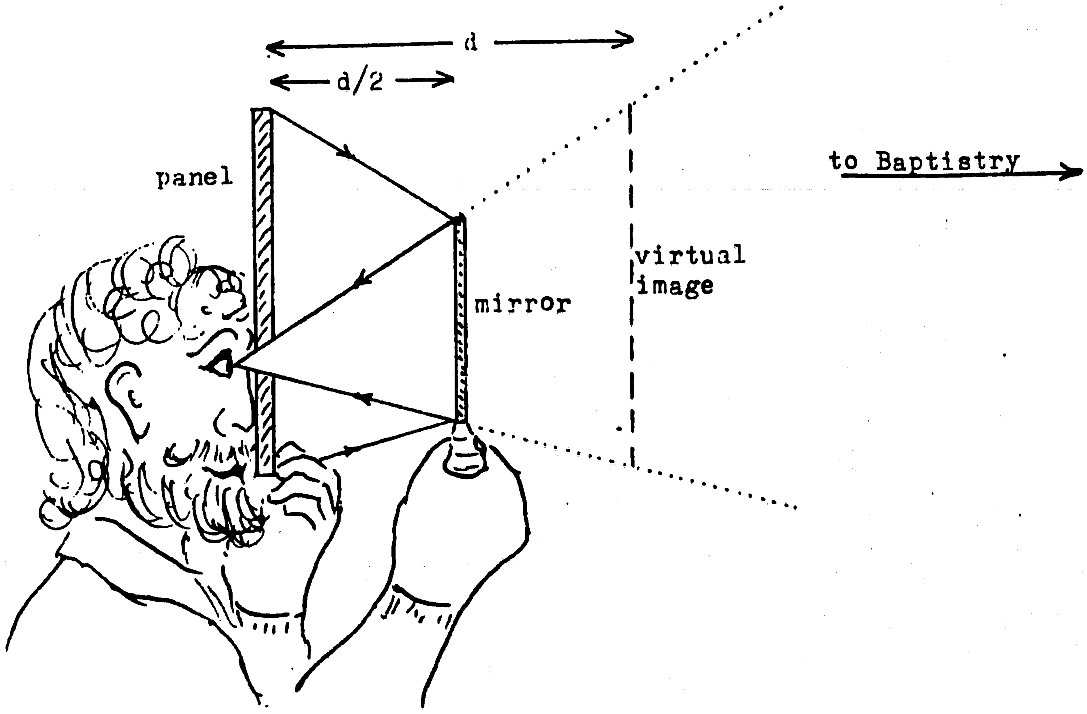
Application 3. Brunelleschi's experiment.

Brunelleschi discovered the rule about vanishing points in about 1420, and explained it to his fellow artists by means of a painting of the Baptistry in Florence. He could not prove he was right *mathematically* because the relevant mathematics had not yet been invented, and so he had to convince them by demonstrating that he was right *visually*. He chose the Baptistry because it was octagonal, and therefore had very pronounced diagonal vanishing points.



He painted it on a square wooden panel of side d, where d was about a foot, or approximately 30cm. He chose a central point C and diagonal points B,D such that BC = CD = d. Therefore by the theorem the viewing distance was also d; everything was beautifully symmetrical.

He then bored a peephole at C, and held a mirror in front of the panel, so that when he looked through the peephole from the back he could see the painting on the front of the panel reflected in the mirror. The diagram below shows the optics. The mirror was size d/2 and held at a distance d/2 from the eye. The viewer therefore saw a virtual image of the panel of size d at a distance d from the eye. Of course what Brunelleschi actually painted was not the Baptistry itself but its reflection, so that it looked correct when viewed in the mirror.



Suppose that the viewer now moved the mirror down a little so that the top of the original Baptistry became visible over the top of the mirror. The top of the image of the panel would correspondingly be cut-off from view. Meanwhile the rest of the image would not have moved because the ray of light from each point on the panel would follow exactly the same path into the eye as it did before, even though it would have been reflected off a different point of the mirror. Therefore, as the mirror was moved slowly downwards, the viewer would see the top of the original growing, and the bottom of the image shrinking, neither of them moving, but the horizontal line separating them would be slowly moving downwards. Consequently the viewer would have been able to verify that each line of the image matched up exactly with its counterpart on the original. Similarly the mirror could be moved slowly from side to side, and hence the accuracy of every line in the painting could have been checked precisely.

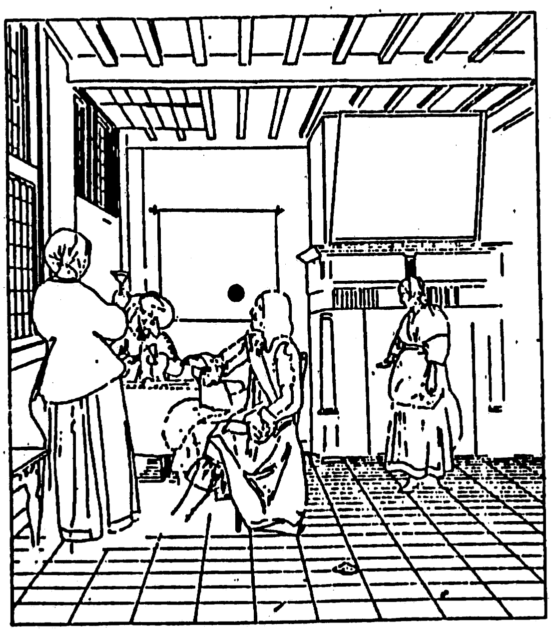
Summarising: there were several advantages to Brunelleschi's experiment. Firstly the novelty caught the viewer's attention. Secondly the mirror movement enabled the accuracy to be verified. Thirdly the peephole ensured that the eye was placed at the correct observation point, and that only one eye was used, thus guaranteeing an enhanced perception of three-dimensionality. Lastly the effect of seeing that enhanced perception through a peephole must have seemed like magic, and must have made a profound psychological impact upon his fellow artists.

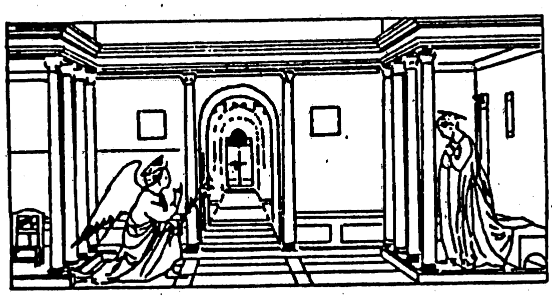
Geometry and perspective

Worksheet 3.

This last worksheet is for you to follow up after seeing the end of the video, to help reinforce what you have learnt. The questions are a mixture of drawing, visual experiments, mathematical proofs, and calculations to find the correct viewing distance to see a picture in perspective. You may find some of the questions a little harder than those on the previous worksheets but do not be deterred, because they all have quite short solutions.

1. The central point C in each of the two sketches of paintings below has been marked with a dot. In the corresponding sketches in Mathematical Notes 3 the radial lines towards C have been drawn in, and a diagonal point D has also been marked. In each case measure the distance d = CD. Make a peephole of height d, place it above C, and verify that the sketch looks 3-dimensional through the peephole. Notice that you get a better perception of 3-dimensionality from the sketches below without the radial lines drawn in.





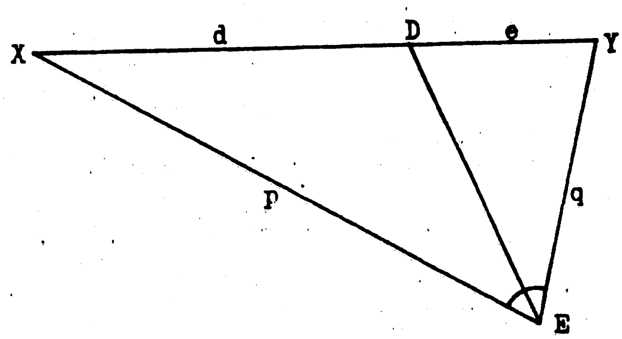
2. Measure the width w of each sketch. Calculate the viewing distances of the original paintings by the formula (W/w)d where

$$W = \text{width of the original} = \begin{cases} 65\text{cm, for de Hoogh's } \textit{Interior} \\ 54\text{cm, for Veneziano's } \textit{Annunciation} \end{cases}$$

The next time you are in London or Cambridge go and see the originals in the National Gallery or the Fitzwilliam Museum. Verify that if you look at the painting with one eye from a point in front of C at the correct viewing distance then you obtain a remarkably enhanced perception of 3-dimensionality. If C is above your eye level ask the museum curator for a stool.

3. If you want to reconstruct Brunelleschi's experiment make the drawing described in Worksheet 2 Question 19 on a piece of white cardboard in a square of side 30cm (rather than 12cm) and scale up all the measurements by a factor of 5/2. Make sure that your drawing is left-right symmetrical. Cut out the square so that you now have a square cardboard panel. Make a photocopy on a piece of paper. After you have made the photocopy drill a peephole in the panel at C with a diameter 2cm. Look at the photocopy through the peephole from the back of the panel at a distance of 30cm, and then interpose a mirror (of any size) halfway in between, as described in Mathematical Notes 3.

The next time you are in Florence take your panel and mirror with you, and, standing in the doorway of the cathedral, try it out on the real thing.



4. Let ED be the angle-bisector of the triangle EXY.

Let DX = d EX = p

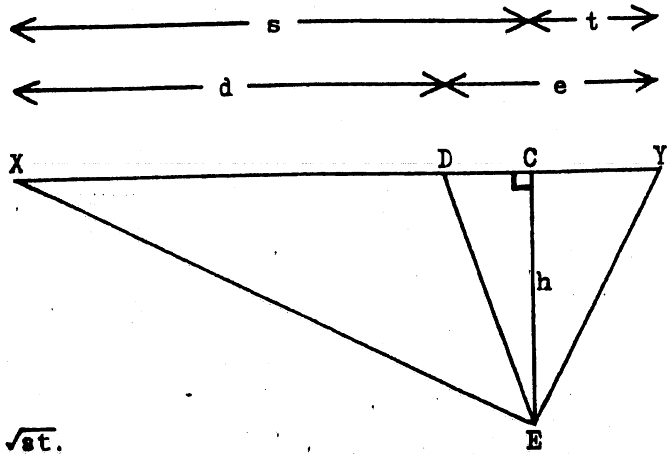
DY = e EY = q

Prove $\frac{p}{q} = \frac{d}{e}$.

[Hint: drop perpendiculars from X,Y to ED and use similar triangles].
Measure the lengths in the diagram and verify that the ratios are equal.

5. Suppose the triangle EXY has a right-angle at E. Let ED be the angle-bisector, and EC the altitude through E.

Let DX = d CX = s
DY = e CY = t
CE = h



Prove $\frac{s}{t} = \left[\frac{d}{e}\right]^2$, and $h = \sqrt{st}$.

6. Suppose that in the last question d = 2e.

Prove that $s = \frac{12e}{5}$, $t = \frac{3e}{5}$, $h = \frac{6e}{5}$.

Calculate s,t,h in the three cases

(i) e = 5 (ii) $e = \frac{20}{3}$ (iii) e = 8 .

Measure the lengths between the points X,D,C,Y in Question 10 below, and in Figures 2 and 3 of Questions 2 and 3 in Worksheet 2. Verify that they agree with you calculations in the three cases.

7. Suppose that 2d = 3e in Question 5.

Prove that $s = \frac{45e}{26}$, $t = \frac{20e}{26}$, $h = \frac{30e}{26}$.

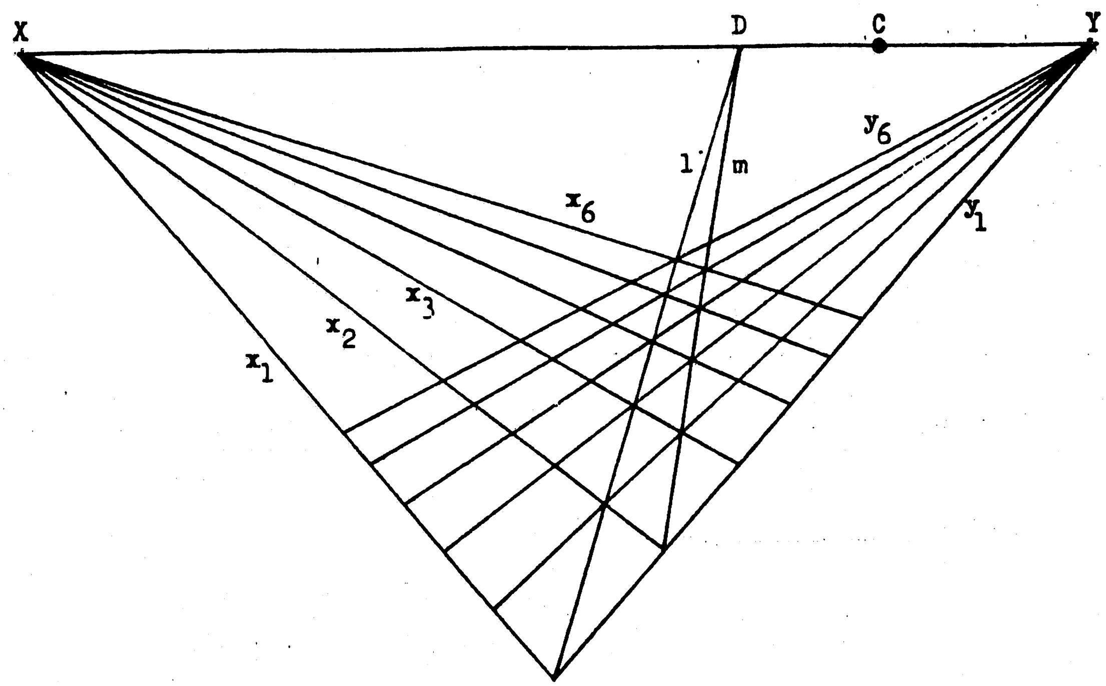
Calculate s,t,h when d = 39, e = 26.

8. In question 5 prove that

$$s = \frac{(d+e)d^2}{d^2+e^2} , \quad t = \frac{(d+e)e^2}{d^2+e^2} , \quad h = \frac{(d+e)de}{d^2+e^2} .$$

9. In Question 5 deduce that if the points X,D,Y are given then the positions of C and E are determined.

10. The diagram is a perspective drawing of a horizontal square-tiled floor, with vanishing points X,Y for the edges of the tiles and vanishing point D for the diagonals of the tiles.



Assuming that the paper is held vertically with XY horizontal, prove there is exactly one observation point and describe its position.

[Hint: use the definition of vanishing point in Mathematical Notes 1, and Questions 5 and 9. Warning: although D is the vanishing point of the diagonals of the floor tiles, it not a "diagonal point" in the sense of Mathematical Notes 3, because the diagonals of the floor tiles are not inclined at 45^o to the central ray].

Deduce that the central point is C, and the viewing distance is 6cm. [Hint : use Question 6.]

Make a peephole of height 6cm; place it above C and verify that the drawing looks in perspective.

11. Draw a horizontal line on the blackboard at eye-level. Mark points XDCY on the line such that XD = 39cm, DC = 6cm, CY = 20cm. Choose two lines ℓ, m through D as in the diagram in the last question, and choose a line x_1 through X. These then determine the whole diagram as follows. Draw in turn the lines $y_1, x_2, y_2, x_3, y_3, \ldots$ according to the rules :

$$y_i = \text{the line through Y and the point where } x_i \text{ meets } \ell,$$
$$x_{i+1} = \text{the line through X and the point where } y_i \text{ meets } m.$$

Draw in the rest of the diagonals and verify that they all go through D.

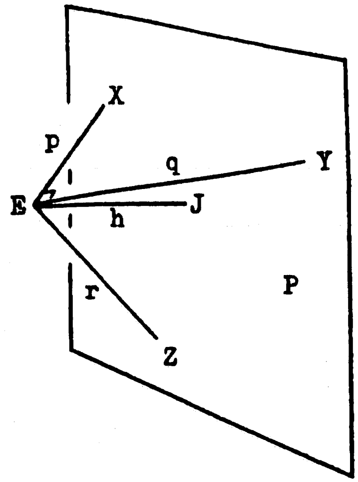
Deduce from Questions 5,7 and 9 that the unique observation point is 30cm in front of C. Place one eye there, close the other eye, and verify that your drawing looks like a square-tiled floor in perspective.

12. In Figure 2 of Question 2 in Worksheet 2 verify that the diagonals of the top faces of the two cubes go through D. Deduce from Questions 5 and 6 above that the unique observation point E is 8cm in front of C. Use Pythagoras' Theorem to calculate EX and EY, and compare your answers with the approximate values for p,q used in Question 15 of Worksheet 2.

13. In Figure 3 of Question 3 in Worksheet 2 verify that the diagonals of the top and bottom faces of the cube go through D. Deduce from Questions 5 and 6 above that the unique observation point (for the cube, as opposed to a rectangular box) is 9.6cm in front of C.

Calculate EX,EY as in the last question and verify that your answers agree with the values of p,q used in Question 16 of Worksheet 2.

14. Let XYZ be an acute-angled triangle in the plane P. Let EX, EY, EZ be at right-angles to each other. Let J be the foot of the perpendicular from E to P (as in Question 8 of Worksheet 2). Let EX = p, EY = q, EZ = r, EJ = h. Prove that



$$h = \frac{1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}} .$$

15. In Question 18 of Worksheet 2 measure the lengths p, q, r of the axes that you made. Calculate h by the last question, and verify that this agrees with the viewing distance from E to the blackboard.

16. In Question 14 let YZ = x, ZX = y, XY = z . Prove that $x^2=q^2+r^2$. Write down similar expressions for y^2 and z^2. Deduce that

$$p^2 = \frac{-x^2 + y^2 + z^2}{2} ,$$

and find similar expressions for q^2 and r^2.

Deduce that

$$h = \frac{1}{\sqrt{\frac{2}{-x^2 + y^2 + z^2} + \frac{2}{x^2 - y^2 + z^2} + \frac{2}{x^2 + y^2 - z^2}}} .$$

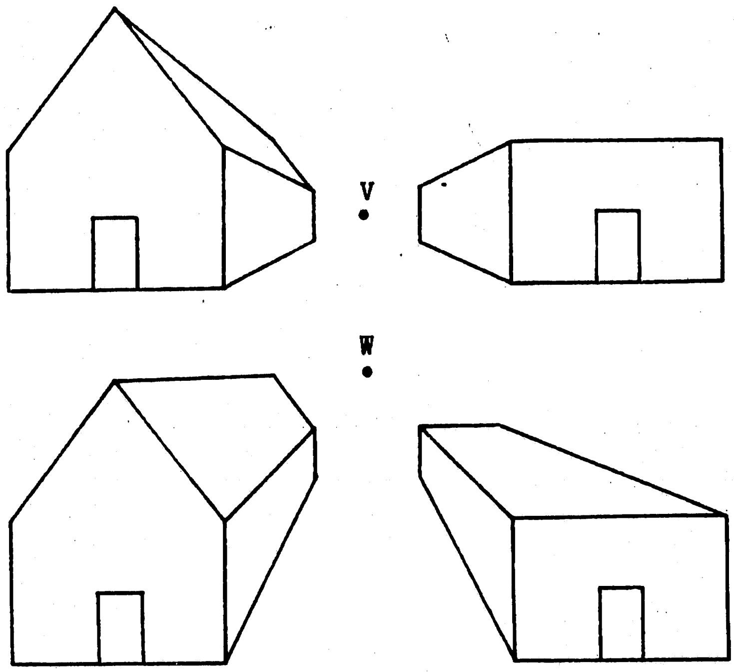
17. In Figure 1 in Question 1 of Worksheet 2 measure the sides x, y, z of the triangle XYZ. Calculate p, q, r by the formulae in the last question, and verify that your answers agree with the values used in Question 14 of Worksheet 2.

Calculate h, and verify that h = 8cm to the nearest millimetre. This explains why the viewing distance for Figure 1 was 8cm.

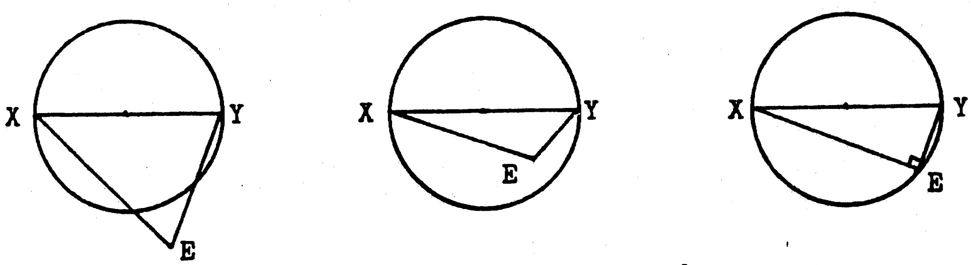
Geometry and perspective

Solutions 1.

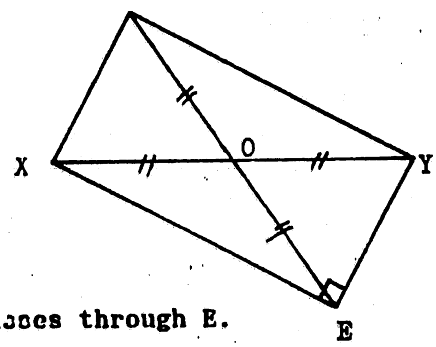
1.



2.



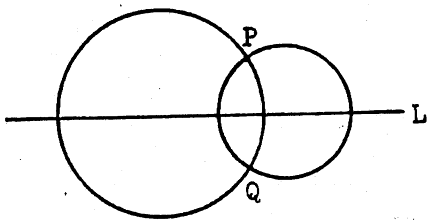
3. Complete the rectangle.
By symmetry the diagonals are equal and bisect each other, at a point O say.
∴ OX = OY = OE.
∴ the circle centre O and radius OX passes through E.
But this __is__ the circle diameter XY.

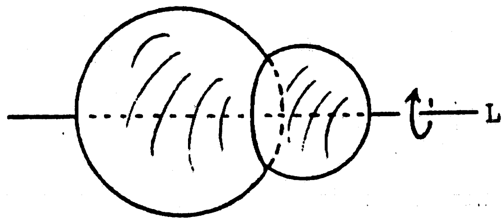


4. Spin the circle about the axis XY to obtain the sphere with diameter XY.

5. Reflection in L maps each circle to itself because L is a diameter of each circle. Therefore the intersection is mapped into itself. In other words the pair of points P, Q is mapped to itself. But P is not mapped to itself because P is not on L. Therefore P is mapped to Q and vice versa.

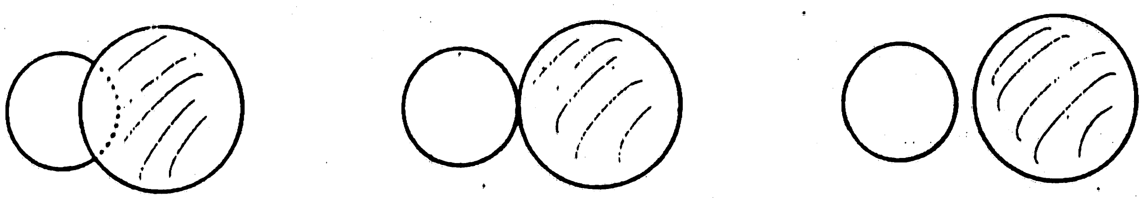
6. Each circle spins into a sphere. The two points P and Q spin into the same circle, which is the intersection of the two spheres.





Conversely given two spheres, consider their intersections with a plane through their centres. The resulting two circles meet, and do not touch, otherwise the spheres would touch. Therefore the circles meet in two points as in question 5. Spinning about L recovers the spheres, and verifies that they intersect in a circle.

7. Either 2 points, 1 point, or no points, according as to whether the circle of intersection of the first two spheres intersects, touches, or misses the third sphere.



Geometry and perspective

Solutions 2.

MEASUREMENTS IN CENTIMETRES THROUGHOUT.

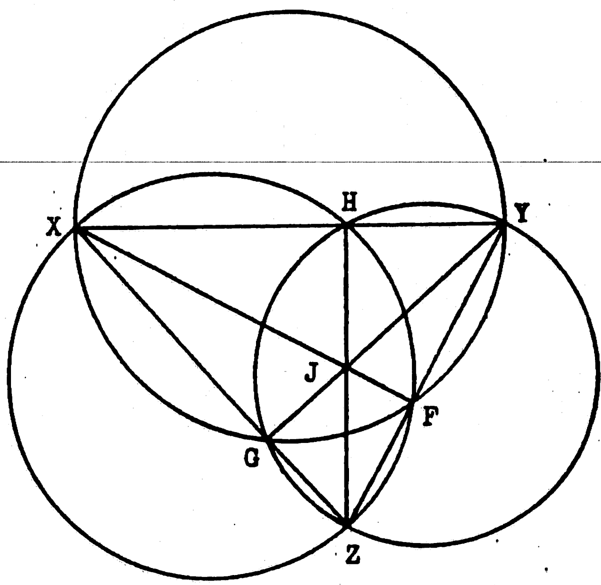
3. XY = 24.

4.

	XC	CY	h = $\sqrt{XC.CY}$
Figure 2	16	4	8
Figure 3	19.2	4.8	9.6

5. In both cases FX,FY are at right-angles. Therefore F lies on the circle with diameter XY (by Lemma 1 of Mathematical Notes 2).

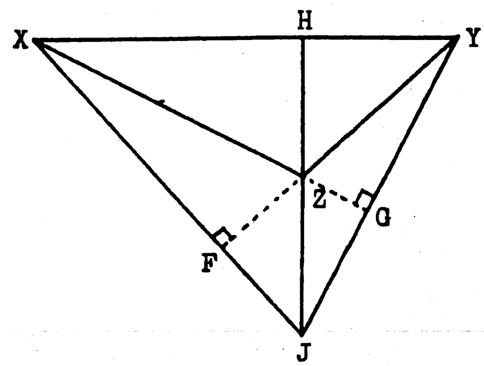
6.



By Question 5 the foot of the altitude through X lies on the circle diameter XY, similarly on the circle diameter XZ, and therefore must be F. In other words XF is the altitude through X. Similarly YG,ZH are altitudes.

__Remark.__ The question does not ask you to __prove__ that the altitudes are concurrent. However the proof in the case of acute-angled triangles follows from the existence of the observation point E (Theorem 1 in Mathematical Notes 2) by Question 8 below. We can extend the result to the case of obtuse-angled triangles as follows:

Suppose XYZ has an obtuse angle at Z. Let J be the meet of the altitudes XF and YG. Let H be the meet of JZ and XY. Now XYJ is an acute-angled triangle, and so does have an orthocentre, which must be Z. Therefore Z lies on the altitude JH of XYJ.



Therefore ZH is the altitude of XYZ though Z.
Therefore the three altitudes of XYZ are concurrent at J.

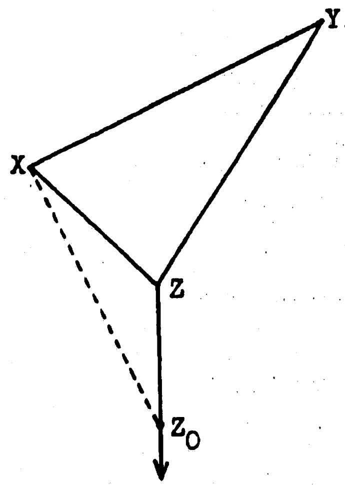
7. Reflection in P maps each sphere to itself, and hence maps their intersection circle C to itself. Therefore since P does not contain C, it must cut C in a diameter. This diameter must be XF, because X and F lie on both spheres by Question 5. Therefore C has diameter XF. Furthermore reflection in P maps the plane containing C to itself, which could not happen unless that plane is perpendicular to P. Therefore C is perpendicular to P.

8. Since E lies on the circle C, it must lie in the plane containing C, which is the plane through the altitude XF perpendicular to P. Therefore the perpendicular from E to P lies in this plane. Therefore the foot J of the perpendicular must lie on XF. Similarly it lies on YG and ZH. Therefore the three altitudes are concurrent at J. In other words J is the orthocentre.

9. Since Z is obtuse F lies outside YZ. Therefore the line through F perpendicular to YZ cannot meet the circle diameter YZ. In other words XF does not meet this circle. Therefore the plane through XF perpendicular to P cannot meet the sphere diameter YZ. Nor can the

circle C meet this sphere, because C lies in that plane by Question 7. Therefore the three spheres with diameter XY,XZ and YZ do not meet. Therefore there cannot be any observation point.

10. As in the proof of Theorem 2 in Mathematical Notes 2, assume that the paper is held so that the edges that are drawn parallel to one another are vertical. Then, as before, any observation point must lie on the horizontal plane through X, and similarly on the horizontal plane through Y. But this time, in contrast to the situation in Theorem 2, these two planes are not the same because XY is not horizontal. Therefore they are parallel, and do not meet. Therefore there is no observation point.



If we start with an acute-angled triangle XYZ then there will be an observation point even through XY is not horizontal, by Theorem 1 of Mathematical Notes 2. If we then let Z descend to minus infinity it will pass through a critical point Z_0 where Z_0X is at right-angles to XY. As Z approaches Z_0 the observation point runs into the paper at X and disappears. When Z has passed below Z_0 there will be no longer an observation point by Question 9, because the triangle will have an obtuse angle at X. Therefore in the limit there will also be no observation point, as we have already proved.

11. Triangles AXE, AOT are similar because corresponding angles are equal, and therefore their sides are proportional.

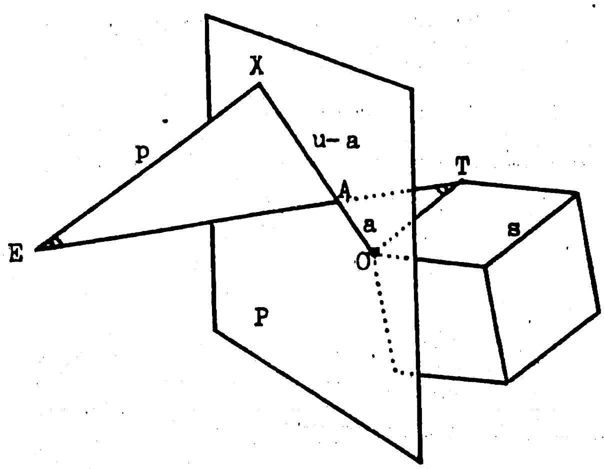
$$\therefore \frac{AX}{AO} = \frac{XE}{OT} \qquad \therefore \frac{u-a}{a} = \frac{p}{s}$$

12. From the last question $\frac{u}{a} - 1 = \frac{p}{s}$

$$\therefore \quad \frac{u}{a} = \frac{p}{s} + 1 = \frac{p+s}{s}$$

$$\therefore \quad \frac{a}{u} = \frac{s}{p+s} \qquad \therefore \quad a = \frac{us}{p+s} .$$

13. Let T be the vertex of the cube whose image is A. Then X is the vanishing point for the edge OT. Therefore EX is parallel to OT by the definition of vanishing point in Mathematical Notes 1.



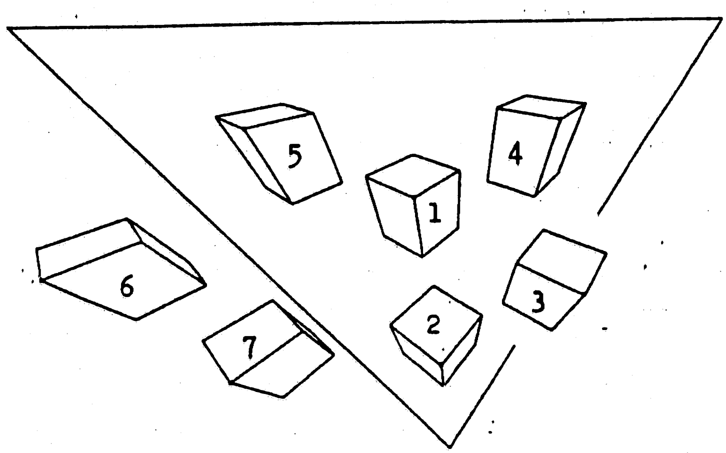
The plane containing EX and OT will also contain A, and the configuration of these five points will be as in Question 11. Hence

$$a = \frac{us}{p+s}$$

by Question 12. The formulae for b,c follow by symmetry.

14. $\frac{s}{p+s} = 0.120, \quad \frac{s}{q+s} = 0.147, \quad \frac{s}{r+s} = 0.234 .$

Label the cubes as in the diagram.



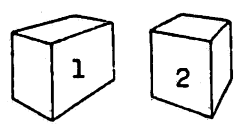
cube	u	v	w	a	b	c
1	17.2	14.0	10.0	2.1	2.1	2.3
2	21.6	17.5	3.3	2.6	2.6	0.8
3	25.0	12.0	8.1	3.0	1.8	1.9
4	22.0	7.4	13.7	2.6	1.1	3.2
5	10.0	19.3	15.0	1.2	2.8	3.5
6	10.0	28.5	17.3	1.2	4.2	4.0
7	16.5	24.0	9.0	2.0	3.5	2.1

Remark. Note that the observation point is determined uniquely by the triangle XYZ (by Theorem 1 of Mathematical Notes 2), and is therefore the same for all the cubes. That is why all the figures look cubical simultaneously from that one point. Furthermore they all look cubical rather than merely rectangular because the lengths of the sides have been drawn according to the formulae in Question 13. The reason why all the cubes look as if they have parallel faces is because they all have the same three vanishing points.

If the sides of the triangle XYZ are known then there are formulae for calculating p,q,r and the height of the peephole. The formulae will be explained in Questions 14, 16 and 17 of Worksheet 3.

15. $\frac{s}{p+s} = \frac{1}{7}$, $\frac{s}{q+s} = \frac{1}{4}$.

Label the cubes as in the diagram.



cube	u	v	a	b
1	11.0	10.0	1.6	2.5
2	17.0	4.6	2.4	1.15

Remark. More accurate values of p,q are p = 17.9, q = 8.9, but this improved accuracy does not affect the values of a,b to the nearest millimeter. An explanation of how to calculate p,q, and of why the observation point is 8cm in front of C, will be given in Questions 5, 6 and 12 of Worksheet 3.

16. u = v = 14 , a = 3.4 , b = 5.5.

Remark. The calculation of p,q and the height and position of the peephole will be given in Question 13 of Worksheet 3.

17. As $w \to \infty$, $\frac{s}{w} \to 0$.

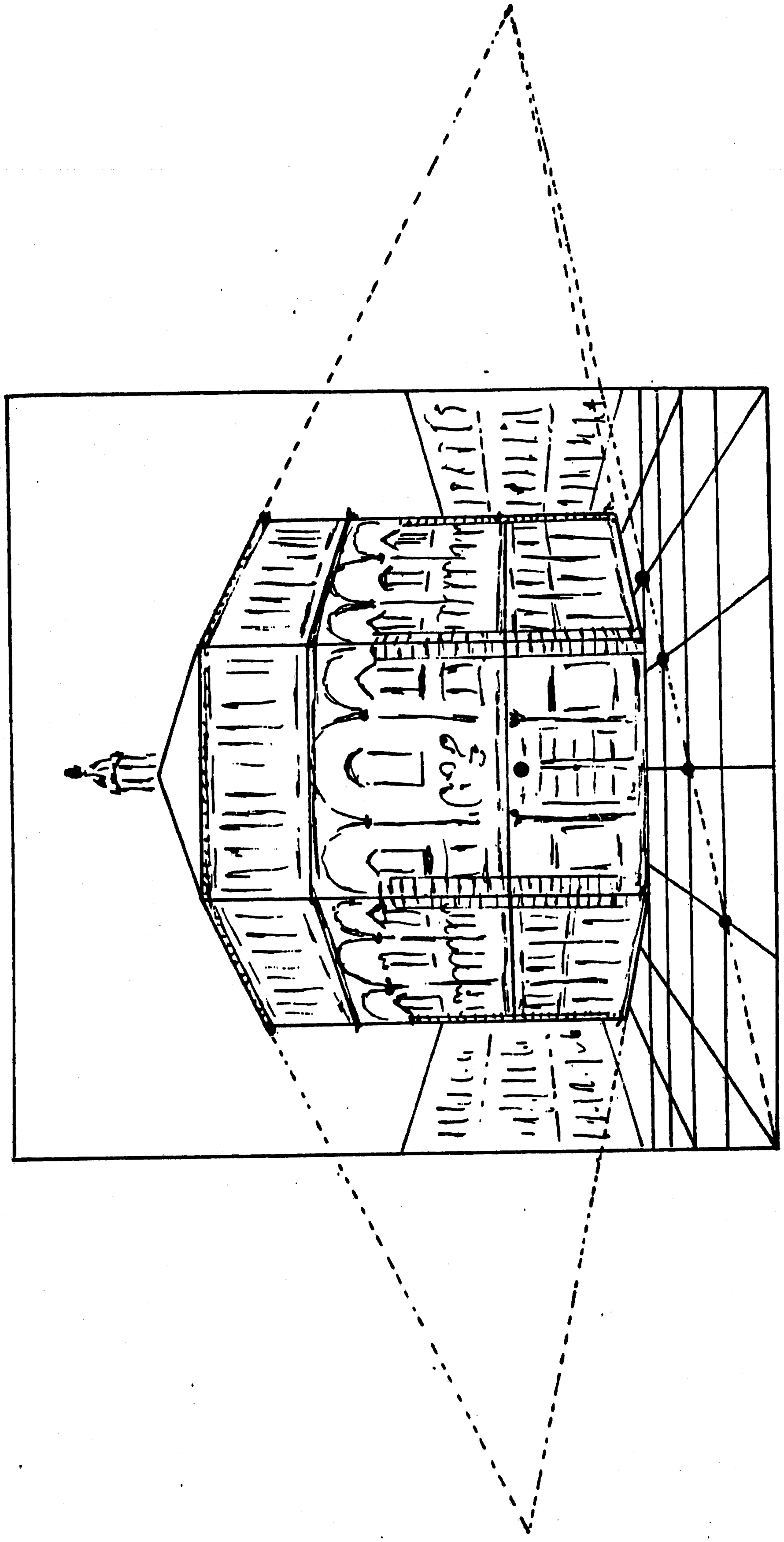
$$\therefore \frac{ws}{r+s} = \frac{s}{\frac{r}{w}+\frac{s}{w}} \to \frac{s}{1+0} = s$$

In Question 13,

$$OC = c = \frac{ws}{r+s} .$$

As Z decends to minus infinity to give Figure 3, the lengths $w,r \to \infty$ in such a way that $\frac{r}{w} \to 1$, and the edge OC of the cube approaches the glass until in the limit it coincides with its image. Therefore $c \to s$.

19.



Geometry and perspective.

Solutions 3.

ALL MEASUREMENTS IN CENTIMETRES.

1. de Hoogh 15, Veneziano 16.

2.

	d	w	W	(W/w)d
de Hoogh	15	7.4	65	132
Veneziano	16	7.4	54	117

Therefore the viewing distances of the original paintings are :

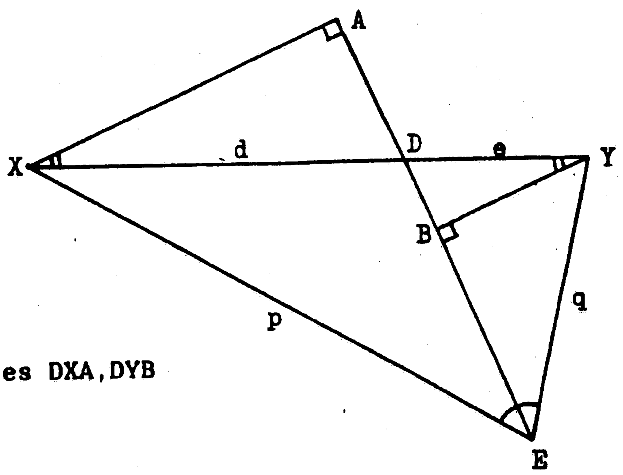
de Hoogh *Interior* 1 metre 32cm,

Veneziano *Annunciation* 1 metre 17cm.

Remark. Notice that in the sketches in Mathematical Notes 3 it is difficult to be very precise when drawing the diagonals of the floor tiles, and so the measurements of d are only accurate to within about 10%. Therefore the calculations above are only accurate to within 10%. In fact, in the video a different reproduction of the de Hoogh painting was used, giving a different estimate for the viewing distance of 1 metre 43cm.

4. Let XA,YB be the perpendiculars from X,Y onto ED.
By similarity of triangles EXA,EYB

$$\frac{p}{q} = \frac{XA}{YB}.$$



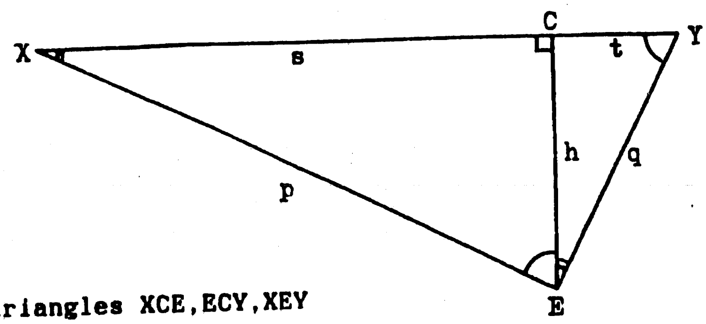
By the similarity of triangles DXA,DYB

$$\frac{XA}{YB} = \frac{d}{e}.$$

$$\therefore \frac{p}{q} = \frac{d}{e}.$$

Measurements: p=8, q=4, d=5.2, e=2.6 . $\therefore \frac{p}{q} = 2 = \frac{d}{e}$.

5. Let EX = p
EY = q .



By similarity of triangles XCE, ECY, XEY

$$\frac{s}{h} = \frac{h}{t} = \frac{p}{q} .$$

$$\therefore \frac{s}{t} = \frac{s}{h} \cdot \frac{h}{t} = \left(\frac{p}{q}\right)^2 = \left(\frac{d}{e}\right)^2 \text{ , by Question 4.}$$

Since $\frac{s}{h} = \frac{h}{t}$, $st = h^2$.

$\therefore$ $h = \sqrt{st}$.

6. $\frac{d}{e} = 2$.

$\therefore$ $\frac{s}{t} = 4$, by Question 5.

$\therefore$ $s = 4t$

$\therefore$ $5t = 4t+t = s+t = XY = d+e = 2e+e = 3e$

$\therefore$ $t = \frac{3e}{5}$. $\therefore$ $s = 4t = \frac{12e}{5}$.

$\therefore$ $st = \frac{36e^2}{25}$. $\therefore$ $h = \sqrt{st} = \frac{6e}{5}$.

Case	e	s	t	h
(i)	5	12	3	6
(ii)	$\frac{20}{3}$	16	4	8
(iii)	8	19.2	4.8	9.6

7. $\frac{d}{e} = \frac{3}{2}$ $\quad\therefore\ \frac{s}{t} = \frac{9}{4}$, by Question 5.

$$\therefore\ 4s = 9t\ .$$

$$\therefore\ 13t = 9t+4t = 4s+4t = 4XY = 4d+4e = 6e+4e = 10e\ .$$

$$\therefore\ t = \frac{10e}{13} = \frac{20e}{26}\ .$$

$$\therefore\ s = \frac{9}{4}\cdot\frac{20e}{26} = \frac{45e}{26}$$

$$\therefore\ st = \frac{900e^2}{26^2}\ . \qquad \therefore\ h = \sqrt{st} = \frac{30e}{26}\ .$$

If $e = 26$ then $s = 45$, $t = 20$, $h = 30$.

8. $\frac{s}{t} = \frac{d^2}{e^2}$, by Question 5.

$$\therefore\ s = \left[\frac{d^2}{e^2}\right]t.$$

Now $s + t = XY = d + e$.

$$\therefore\ \left[\frac{d^2}{e^2}\right]t + t = d + e\ .$$

$$\therefore\ \left[\frac{d^2+e^2}{e^2}\right]t = d + e$$

$$\therefore\ t = \frac{(d+e)e^2}{d^2+e^2}\ .$$

$$\therefore\ s = \left[\frac{d^2}{e^2}\right]t = \frac{(d+e)d^2}{d^2+e^2}\ .$$

$$\therefore\ st = \frac{(d+e)^2d^2e^2}{(d^2+e^2)^2}$$

$$\therefore\ h = \sqrt{st} = \frac{(d+e)de}{d^2+e^2}$$

9. Given X,D,Y, measure d and e, calculate s as above, and hence mark the point C. Next calculate h, draw the line through C perpendicular to XY, and mark E on the line a distance h below C.

10. By the definition of vanishing points EX,EY are parallel to the edges of the floor tiles. Therefore they are horizontal and at right-angles. Therefore the triangle EXY is horizontal and has a right-angle at E,

Meanwhile ED is parallel to the diagonals of the tiles, and is therefore inclined at 45° to both EX and EY. In other words ED is the angle-bisector of the triangle EXY. Therefore the triangle is as in

Question 5. The positions of X,D,Y determine the positions of C and E by Question 9. Therefore there is exactly one observation point.

Since EC is perpendicular to XY and horizontal, it is perpendicular to the paper. Therefore C is the central point, and the viewing distance = EC = h. The numerical values are given by Question 6(i):

$$d = 10 \,,\quad e = 5 \,,\quad s = 12 \,,\quad t = 3 \,,\ h = 6 \;.$$

Therefore E is 6cm in front of C.

11. Same argument as in Question 10.
 The numerical values are given by Question 7 :

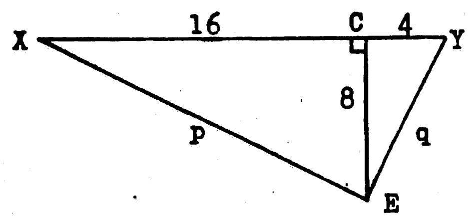
$$d = 39 \,,\quad e = 26 \,,\ s = 45 \,,\ t = 20 \,,\quad h = 30 \;.$$

 Therefore E is 30cm in front of C.

12. Same argument as in Question 10.
 The numerical values are given by Question 6(ii) :

$$d = \frac{40}{3} \,,\quad e = \frac{20}{3} \,,\quad s = 16 \,,\quad t = 4, \quad h = 8 \;.$$

 Therefore E is 8cm in front of C.



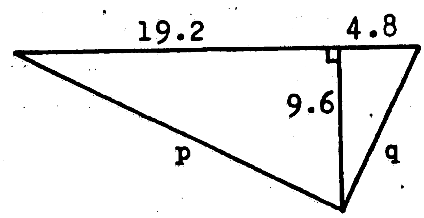
$$\therefore \quad p = EX = \sqrt{16^2+8^2} = 8\sqrt{5} = 17.9$$
$$q = EY = \sqrt{8^2+4^2} = 4\sqrt{5} = 8.9$$

 These are within 1 millimetre of the values p = 18, q = 9 used in Question 15 of Worksheet 2.

13. Same argument as in Question 10.
 The numerical values are given by Question 6(iii) :

$$d = 16 \,,\ e = 8 \,,\quad s = 19.2 \,,\quad t = 4.8 \,,\quad h = 9.6$$

 Therefore E is 9.6cm in front of C.
 By Pythagoras:



$$p = 21.5 \,,\quad q = 10.7 \;.$$

14. Take coordinates (ξ,η,ζ) with respect to origin E and axes EX,EY,EZ.

The equation of the plane P is :

$$\frac{\xi}{p} + \frac{\eta}{q} + \frac{\zeta}{r} - 1 = 0.$$

Therefore the distance from the point (ξ,η,ζ) to P is :

$$\frac{\left|\frac{\xi}{p} + \frac{\eta}{q} + \frac{\zeta}{r} - 1\right|}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}}$$

Put $\xi = \eta = \zeta = 0$ to obtain the desired result.

16. By Pythagoras, $x^2 = q^2 + r^2$

By symmetry, $y^2 = r^2 + p^2$

$z^2 = p^2 + q^2$

$\therefore\ -x^2 + y^2 + z^2 = 2p^2$

$\therefore\ p^2 = \dfrac{-x^2+y^2+z^2}{2}$

By symmetry, $q^2 = \dfrac{x^2-y^2+z^2}{2}$

$r^2 = \dfrac{x^2+y^2-z^2}{2}$

Substitute these in the formula for h in Question 14.

17. $x = 20.0$ $\quad\therefore\ p = 21.9$ $\quad\therefore\ h = 8.0$

$y = 24.0$ $\quad q = 17.4$

$z = 28.0$ $\quad r = 9.8$