CLASSICAL GEOMETRIES

11. Projections and collineations

11.1 Projections

We have already seen an example of a projection, an artist drawing a picture in the picture plane of an object plane. A central property of projections is that the projection of a line is a line. This brings up some questions: Are there any other correspondences that take lines to lines? Can we describe such correspondences efficiently? What are some basic properties? Another reason to look at such correspondences is to be able to understand what it means for two projective planes to be the "same."

Suppose that we have two projective planes Π_1 and Π_2. Let f be a one-to-one, onto function from the points of Π_1 to the points of Π_2. We write this as

$$f : \Pi_1 \to \Pi_2$$

We say that f is a *collineation* if for any line in Π_1 the image under f is a line in Π_2. For example, a projection or a composition of projections, which is called a *projectivity*, is a collineation. Figure 11.1.1 shows a projectivity from the picture plane to itself.

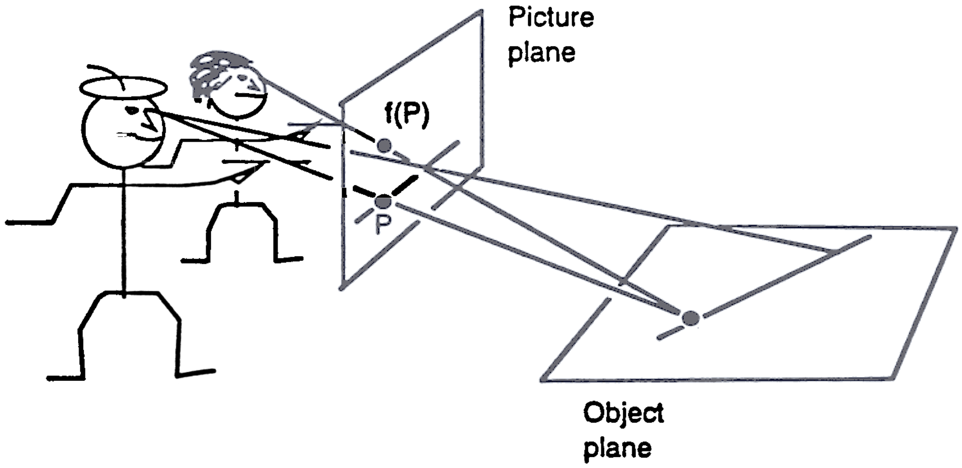

Figure 11.1.1

Suppose that we have a projective plane. We can label that plane differently or create another copy somewhere else, and it will be essentially the same projective plane. We need a language to be able to say that two projective planes are "essentially" the same. We put that as follows: Let Π_1 and Π_2 be two projective planes. We say that they are *equivalent* if there is a collineation from one to the other. For example, the (extended) picture plane and the (extended) object plane are equivalent projective planes. In fact, any two (extended) planes in 3-space are equivalent.

11.2 Affine functions and homogeneous coordinates

Suppose that we have a collineation of a projective plane to itself. How can we describe it efficiently? Can we find them all? For example take the extended real plane. Translations, rotations, reflections, dilations, and shears are all examples of functions that can be extended to a collineation of the extended real plane. In fact, any composition of a non-singular linear function and a translation, which is called an *Affine linear function* (or an *Affinity* is an old-fashioned word for this), can be extended to a collineation of the extended real plane. We describé such an Affine linear function from the plane to itself as follows:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + b_1 \\ a_{21}x + a_{22}y + b_2 \end{pmatrix}$$

or more compactly as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = A_0\begin{pmatrix} x \\ y \end{pmatrix} + B_0,$$

where A_0 is a non-singular 2 by 2 matrix, and B_0 is a 1 by 2 column matrix.

Note that for such Affine linear functions, the image of any point in the line at infinity is again a (possibly different), point in the line at infinity. In homogeneous coordinates, we get the following expression for an Affine linear function:

$$f\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} A_0 & B_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

where we write the 3 by 3 matrix for f in terms of smaller blocks.

Affine linear functions are good for the Affine plane, but they leave something to be desired for the projective plane. In particular, there are many collineations that are not Affine linear functions. We use homogeneous coordinates to describe these other collineations. Let A be any non-singular 3 by 3 matrix. We represent points in a projective plane Π by homogeneous coordinates. Then a collineation $f : \Pi \to \Pi$. is given by:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

It is easy to check that this is well-defined and a collineation. Affine linear functions are a special case. Figure 11.2.1 graphically shows this more general collineation by showing the image of the Affine plane $z = 1$.

11.3 Moving points around

How much extra freedom does this new kind of collineation buy you? It is not hard to see that any three non-collinear points in the Affine plane can be transformed into any other three by an Affine linear function. Can we do better in a projective plane? Let Π be any projective plane defined using some field $\mathbf{F}$.

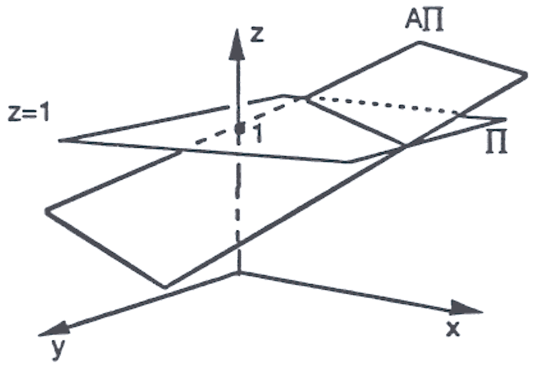

Figure 11.2.1

Theorem 11.3.1. *In the projective plane Π let* $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$, *and* $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$, *be two (labeled) sets of 4 points, where in each set no three are collinear. Then there is a collineation* $f : \Pi \to \Pi$, *given by a matrix as in section 11.2, such that* $f(\mathbf{p}_i) = \mathbf{q})i$ *for* $i = 1, 2, 3, 4$.

Proof. We use homogeneous coordinates. Let $(\mathbf{p}_i)$ and $(\mathbf{q}_i)$ be non-zero vectors in $\mathbf{F}^3$ representing $\mathbf{p}_i$ and $\mathbf{q}_i$ respectively. We must be careful not to confuse the vector $(\mathbf{p})$ with the equivalence class determined by $(\mathbf{p})$, which can be identified with $(\mathbf{p})$, a point in the projective plane. Recall that $t(\mathbf{p}), t \neq 0$, t in $\mathbf{F}$, represents the same point as $(\mathbf{p})$ in the associated projective plane. Choose any three non-zero scalars t_1, t_2, t_3, in $\mathbf{F}$. Then define the 3 by 3 matrix A by

$$A(\mathbf{p}_i) = t_i(\mathbf{q}_i), \text{ for } i = 1, 2, 3.$$

We know that A exists, since the three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, do not lie on a line in the projective plane through the origin in $\mathbf{F}^3$, and thus they are linearly independent in $\mathbf{F}^3$.

Our next task is to choose the t_i's so that the fourth projective point has the desired image. We notice that the three vectors $(\mathbf{p}_1), (\mathbf{p}_2), (\mathbf{p}_3)$ span $\mathbf{F}^3$. Hence there are scalars $a_i \neq 0$, in $\mathbf{F}, i = 1, 2, 3$, such that

$$a_1(\mathbf{p}_1) + a_2(\mathbf{p}_2) + a_3(\mathbf{p}_3) = (\mathbf{p}_4).$$

If one of the scalars a_i were 0, then the other three vectors would be dependent, and the corresponding points in the projective plane would be collinear. Thus $a_i \neq 0, i = 1, 2, 3$ as claimed. A similar argument shows that there are scalars $b_i \neq 0$, in $\mathbf{F}, i = 1, 2, 3$, such that

$$b_1(\mathbf{q}_1) + b_2(\mathbf{q}_2) + b_3(\mathbf{q}_3) = (\mathbf{q}_4).$$

Thus

$$\begin{aligned} A(\mathbf{p}_4) &= A(a_1(\mathbf{p}_1) + a_2(\mathbf{p}_2) + a_3(\mathbf{p}_3)) \\ &= a_1 A(\mathbf{p}_1) + a_2 A(\mathbf{p}_2) + a_3 A(\mathbf{p}_3) \\ &= a_1 t_1(\mathbf{q}_1) + a_2 t_2(\mathbf{q}_2) + a_3 t_3(\mathbf{q}_3). \end{aligned}$$

Now it is clear that we should take $t_i = b_i/a_i, i = 1, 2, 3$. Then

$$A(\mathbf{p}_4) = b_1(\mathbf{q}_1) + b_2(\mathbf{q}_2) + b_3(\mathbf{q}_3) = (\mathbf{q}_4).$$

So we define $f(\mathbf{p})$ to be the line through $A(\mathbf{p})$, for every point $\mathbf{p}$ in the projective plane. This is what was desired and finishes the proof.

11.4 More collineations

Have we found all the collineations of a projective plane? Of course we assume that our projective plane comes from a field, but even then the answer to our question depends on which field.

For any field $\mathbf{F}$ suppose that we have a one-to-one onto function $f : \mathbf{F} \to \mathbf{F}$ defined. We say that f is a field automorphism if the following two properties hold for all z, w in $\mathbf{F}$:

$$f(z + w) = f(z) + f(w)$$

$$f(zw) = f(z)f(w)$$

It is easy to check that $f(1) = 1$, and $f(0) = 0$, and the inverse function f^{-1} is an automorphism.

For example, take the complex numbers $\mathbf{C}$ as our field. (It turns out that the real numbers are not as well suited.) In Chapter 8 we saw that the complex numbers had complex conjugation defined. We define $f(z) = \bar{z}$, the complex conjugate of z, and it is clear that f is a field automorphism that is not the identity.

If we have collineation that is defined as in Section 11.3 and it fixes 4 points, no three collinear, then it must be the identity. Let us take homogeneous coordinates to describe our projective plane. Define a collineation $f : \Pi \to \Pi$ as follows:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(x) \\ f(y) \\ f(z) \end{pmatrix}$$

It is easy to check that f is a well-defined function on the points of the projective plane.

We first show that f is a collineation. Suppose that $[A, B, C]$ defines a line in homogeneous coordinates as in Chapter 9, Section 9.3. Then if

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is any point lieing on that line it satisfies

$$[A, B, C]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ax + By + Cz = 0,$$

and we see that

$$[f(A), f(B), f(C)]\begin{pmatrix} f(x) \\ f(y) \\ f(z) \end{pmatrix} = f(A)f(x) + f(B)f(y) + f(C)f(z)$$
$$= f(Ax + By + Cz)$$
$$= 0$$

Hence we see that the image of the points incident to a line $[A, B, C]$ are the points incident to the line $[f(A), f(B), f(C)]$. So f defines a collineation.

Notice that any point, which has only 0 or 1 coordinates in homogeneous coordinates, is fixed by f. Thus the following four points are fixed:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and it is easy to check that no three of these points are colinear. Thus f, as we have just defined it , does not come from a 3 by 3 matrix we defined it in Section 11.2. We have at least two different types of collineations of a projective plane.

The following Theorem tells us that there are no more collineations, at least for projective planes defined over a field. This is sometimes called the Fundamental Theorem of projective geometry. We will mention more about the proof in later chapters but not right now.

Theorem 11.4.1. *Let f be any collineation of a projective plane defined over a field. Then f is the composition $f = f_1 f_2$ of two collineations, where f_1 is defined by a 3 by 3 matrix (as in Section 11.2, and called a projectivity), and f_2 is defined by a field automorphism as above.*

Exercises:

1. In Figure 11.1.1 a collineation is shown. In the Affine coordinates of the picture plane what "kind" of collineation is it? Be as specific as possible.
2. Suppose that a collineation f of a projective plane Π to itself is obtained, in a 3-dimensional projective space over a field, as the composition of projections of various projective planes starting and ending at Π. Show that f comes from a 3 by 3 matrix as in Section 11.2.
3. Show that any collineation coming from a 3 by 3 matrix as in Section 11.2 can be written as the composition of collineations of the same sort each of which fix all the points on some line, a possibly different line for each collineation. You may use the proof of Theorem 11.3.1 or what you may know about linear algebra (as long it is correct).
4. Show that any collineation of projective plane to itself, as in Section 11.2, can be written as the composition of projections as in Exercise 2 above.

5. Show that any projectivity of a projective plane over the real field fixes some point and for some line takes all the points on that line to points on the same line.
6. Suppose that we have two sets of three labelled distinct points all incident to a single line in any projective plane. Show that there is a composition of projections that take the one labelled set onto the other. What is the least number of projections needed?
7. Consider the following subset of the real numbers:

$$\mathbf{F} = \{a + b\sqrt{2} \quad a \text{ and } b \text{ rational}\}.$$

We have already shown that $\mathbf{F}$ is a field. Find a field automorphism that is not the identity.
8. Let f be a field automorphism of the real numbers.

   a. Show that if $x > 0$, then $f(x) > 0$.
   b. Show that if $x < y$, then $f(x) < f(y)$.
   c. Show that if x is rational, then $f(x) = x$.
   d. Show that f is the identity. Hence every collineation of the real projective plane is a projectivity.