

CLASSICAL GEOMETRIES

12. Duality and polarity

In previous sections we have seen the definition of an abstract projective plane, what it means for two planes to be equivalent, and at least two examples of seemingly different but equivalent projective planes, the extended Euclidean plane and the definition using homogeneous coordinates. Here we review some of these “models” for the projective plane and use those descriptions to define a “polarity”, which is a kind of self-equivalence that interchanges points and lines.

12.1 Models of the projective plane

Recall that for the definition of a projective plane, we did not require that incidence of a point and line necessarily be that the point was an element of the line. If we want to insist, we *could* redefine all the lines so that each line is the collection of points that are incident to it, but in some models, that will come up shortly, it is convenient to allow ourselves the freedom of a more general incidence relation.

Recall that our first model of the projective plane was the Extended Euclidean Plane. Once we created our ideal points and line at infinity, incidence was defined as “being a member of.”

In Chapter 9 we defined another model for “the” projective plane using homogeneous coordinates in some field. A projective point was defined as a line through the origin in the 3-dimensional vector space over the field. A projective line was defined as a plane in the vector space. Incidence was defined as set containment. Let us call this model the Homogeneous Model 1.

We can alter this definition slightly and say that a projective point is the set of vectors on a line through the origin, except for the origin itself. We say that a projective line is a line through the origin with the origin removed. The line in the vector space corresponds to the $[A, B, C]$ vector that makes up the coefficients of the equation that defines the plane in the Homogeneous Model 1. Thus incidence between a projective point and a projective line is perpendicularity between the corresponding lines in the vector space. This definition has the advantage that points and lines are treated more equally, and the language of equivalence classes on the non-zero vectors in the 3-dimensional vector space can be used. We call this model the Homogeneous Model 2.

In order to get a better understanding of more of the geometry than the incidence structure, we can alter the above models even further in the case of real projective geometry. We can intersect the unit sphere in Euclidean 3-space with each of the classes in the Homogeneous models 1 and 2. We say that two points p and q on the unit sphere in real 3-space are *antipodal* if $q = -p$. So two distinct points on the unit sphere are antipodal if

and only if they lie on a line through the origin. We say that a *great circle* is the intersection of a plane through the origin with the unit sphere.

We now define the Sphere Model 1 for the real projective plane. Here a projective point is a pair of antipodal points on the real unit sphere. A projective line is a great circle on the unit sphere. Incidence is just set containment. Clearly this model is equivalent to the Homogeneous Model 1 over the real field. See Figure 12.1.1

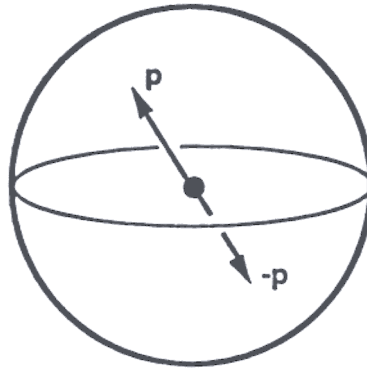


Figure 12.1.1

Finally we define the Sphere Model 2 for the real projective plane. Again a projective point is defined to be a pair of antipodal points. However, a projective line is defined to be a pair of antipodal points. Incidence is perpendicularity.

We summarize this in the following table.

The Plane	Points	Lines	Incidence
Extended Euclidean Plane	Finite points & ideal points at infinity	Ordinary points & the line at infinity	"is a member of"
Homogeneous coordinates 1	Lines through 0	Planes through 0	Set containment
Homogeneous coordinates 2	Lines through 0 except 0	Lines through 0 except 0	Perpendicularity
Sphere model 1	Antipodal points on the sphere	Great circles on the sphere	Set containment
Sphere model 2	Antipodal points on the sphere	Antipodal points on the sphere	Perpendicularity

It is a good exercise to convince yourself that all of these models for the real projective plane are equivalent.

12.2 Polarities

Recall that our axioms for a projective plane had a symmetry with respect points and lines. In fact, the word “point” and the word “line” can be interchanged and the axioms remain the same. Any statement in projective geometry can reverse the words “line” and “point” and it should make sense. But will one statement be true when the other is? If the truth of one statement follows directly from the original three axioms, the answer will be yes. But some of the statements that we proved, for example Pascal’s Theorem, relied on using more than our three axioms. So we look for a more explicit way of relating the points and lines.

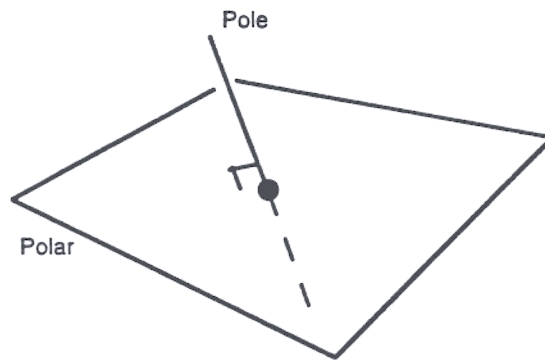


Figure 12.2.1

Recall from Chapter 11 that a collineation between projective planes was an incidence preserving function taking points to points and lines to lines. Here we define a similar notion but with points and lines being interchanged. Let π be any projective plane and let

$$\alpha : \{\text{points of } \pi\} \rightarrow \{\text{lines of } \pi\}$$

be a one-to-one onto function such that for any point p of π and any line l of π , p and l are incident if and only if $\alpha(p)$ and $\alpha(l)$ are incident, where we use α to denote the inverse of α . Such a function is called a *polarity*. For a point p we call $\alpha(p)$ the *polar* of p , and for a line l we call $\alpha(l)$ the *pole* of l . Note that the existence of such a polarity says that, in a certain sense, points and lines are interchangeable.

We present a few examples. Let us take the Homogeneous Coordinates Model 1. We can take the polar of a line through the origin to be the plane through the origin perpendicular to that line, and so the pole of plane is the line through the origin perpendicular to it.

For the Sphere Model 1, the polar of a pair of antipodal points is the great circle equidistant to them, and the pole of a great circle is the pair of antipodal points that are on the line through the origin perpendicular to the plane of the great circle. So the pole of

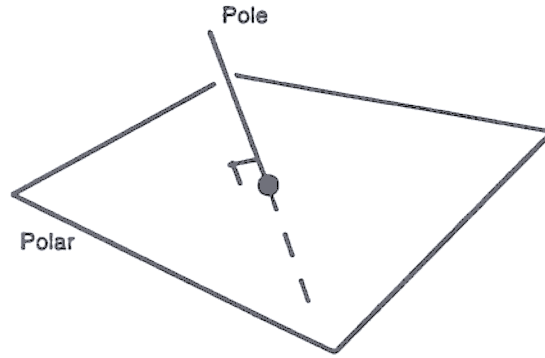


Figure 12.2.2

the equator on our Earth are the North and South Poles, which is the reason for its name. See Figure 12.2.2.

For the Homogeneous Model 2 and the Sphere Model 2, the polarity takes on a very simple form. Recall that points in the projective plane were equivalence classes of non-zero column vectors and lines were non-zero row vectors. The polarity is given as follows:

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = [x, y,$$

It is easy to check that this is a polarity and is the same as the other polarities mentioned before.

12.3 More polarities

One unfortunate (or possibly just distinctive) property of the polarity mentioned above is that a projective point is never incident to its polar. This kind of polarity is called an *elliptic* polarity. Otherwise the polarity is called *hyperbolic*. Note that a polarity followed by a collineation is again a polarity. Thus the following function α is seen to be a polarity.

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = [x, y,$$

Indeed, a point is incident to its polar if and only if

$$0 = \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = [x, y, -z] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 - z^2.$$

So a point is incident to its polar if and only if it lies on a conic, the circle $x^2 + y^2 = 1$ in the Affine plane $z = 1$. Thus α is a hyperbolic polarity.

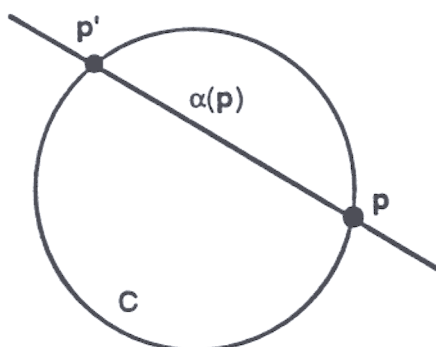


Figure 12.3.1

For this polarity we now give a very simple geometric description using the circle C defined above.

First, we claim that for any point p in C , $\alpha(p)$ is tangent to C . If not, then $\alpha(p)$ is incident to another point p' in C as in Figure 12.3.1. Then $\alpha(\alpha(p)) = p$ is incident to $\alpha(p')$; and since p' is on C , $\alpha(p')$ is incident to p' as well. Thus $\alpha(p') = \alpha(p)$, which contradicts that α is one-to-one. Thus $\alpha(p)$ is tangent to C .

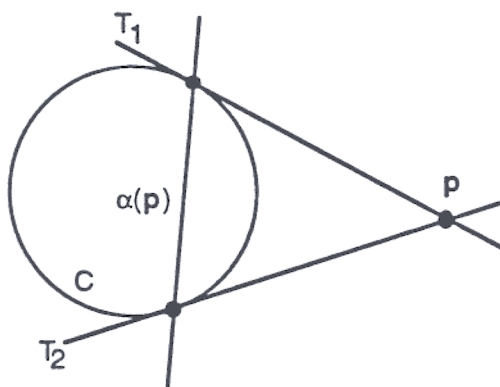


Figure 12.3.2

Suppose that p is a point outside the circle C . Let T_1 and T_2 be the two tangents to C that are incident to p . See Figure 12.3.2. So $\alpha(T_1)$ and $\alpha(T_2)$ are the points of tangency of T_1 and T_2 , respectively, on C , and $\alpha(T_1)$ and $\alpha(T_2)$ are both incident to $\alpha(p)$. Thus $\alpha(p)$ is the line determined by the two points of tangency of T_1 and T_2 .

Suppose that p is a point inside the circle C . Choose two distinct lines l_1 and l_2 that are incident to p . Reverse the construction of Figure 12.3.2 to find the poles $p_1 = \alpha(l_1)$ and $p_2 = \alpha(l_2)$. Then p_1 and p_2 are incident to $\alpha(p)$, and so they determine the polar $\alpha(p)$, and so they determine the polar $\alpha(p)$. See Figure 12.3.3.

We should keep in mind that the above construction must be correct because we know from Section 12.2 that such a hyperbolic polarity exists. It turns out that we could have chosen any conic to replace C in the above construction.

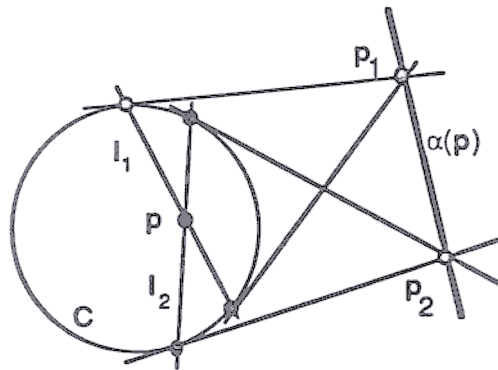


Figure 12.3.3

One consequence of there being polarities is that statements about point conics can be dualized to statements about line conics. For example, Pascal's Theorem becomes the statement of Briachon's Theorem. The polar of a point conic C about C is the associated line conic for C .

Exercises:

1. Suppose that a point $\mathbf{p}$ in the Extended Euclidean plane has Affine coordinates

$$\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Describe the polar of Section 12.2 and the polar of Section 12.3 of $\mathbf{p}$ in terms of the equation of the line $\alpha(\mathbf{p})$ in Affine coordinates.

2. Consider the unit circle C and the polarity α of Section 12.3 of $\mathbf{p}$ in the Affine plane
 - a. Show that if $\mathbf{p}$ is not the center of C , then the line through the center of C and $\mathbf{p}$ is perpendicular to $\alpha(\mathbf{p})$.
 - b. Suppose that the distance from the center of C to $\mathbf{p}$ is t . What is the distance from the center of C to the line $\alpha(\mathbf{p})$?
3. For the polarity of Problem 2 above, what is the image of a point circle, other than C ?
4. For the polarity of Problem 2 above, the image of the following point ellipse is a line ellipse.

$$(x/a)^2 + (y/b)^2 = 1.$$

What are the major and minor axes of this line ellipse?

5. Consider the following model. Abstract points are the points inside the unit circle in the Euclidean plane as well as pairs of antipodal points on the boundary. An

abstract line is defined as any line segment or circular arc between antipodal points on the boundary. See Figure 12.E.1.

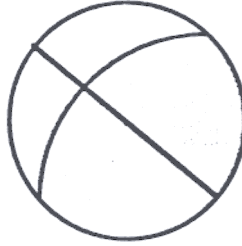


Figure 12.E.1

Show that this model satisfies the axioms for a projective plane.

6. Recall, in Chapter 8, our construction of the skew field $\mathbf{H}$, the quaternions. Elements of $\mathbf{H}$ were defined as certain 2 by 2 matrices with entries in the complex field. We say that a one-to-one, onto function $f : \mathbf{H} \rightarrow \mathbf{H}$ is an anti-automorphism if for every x and y in $\mathbf{H}$,

$$f(x + y) = f(x) + f(y)$$

and

$$f(xy) = f(y)f(x)$$

- a. Show that the function defined on $\mathbf{H}$ which takes a matrix to its transpose is a well-defined anti-automorphism of $\mathbf{H}$.
- b. Suppose that f is any anti-automorphism of $\mathbf{H}$ such as in part a, for example. Consider the projective plane over $\mathbf{H}$ using homogeneous coordinates. Show that the following function α is a well-defined polarity for this projective plane.

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = [f(x), f(y), f(z)].$$