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Math 452, Spring 2008

CLASSICAL GEOMETRIES

14. The geometry of circles

So far we have been studying lines and conics in the Euclidean plane. What about circles, one of the basic objects of study in Euclidean geometry? One approach is to use the complex numbers **C**. Recall that the projectivities of the projective plane over **C**, which we call $\mathbf{CP}^2$, are given by 3 by 3 matrices, and these projectivities restricted to a complex projective line, which we call a $\mathbf{CP}^1$, are the Moebius functions, which themselves correspond to a 2 by 2 matrix. The Moebius functions preserve the cross ratio. This is where circles come in.

14.1 The cross ratio for the complex field

We look for another geometric interpretation of the cross ratio for the complex field, or better yet for $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$. Recall the polar decomposition of a complex number $z = re^{i\theta}$, where $r = |z|$ is the magnitude of z, and θ is the angle that the line through 0 and z makes with the real axis. See Figure 14.1.1.

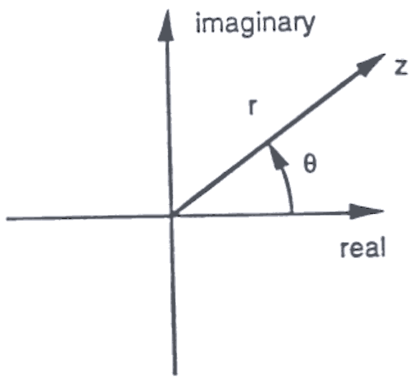

Figure 14.1.1

So if we have 3 complex numbers z_1, z_2, and z_4, the polar decomposition of the ratio

$$\frac{z_1 - z_2}{z_1 - z_4} = \left|\frac{z_1 - z_2}{z_1 - z_4}\right| e^{i\theta_1}$$

the angle θ_1 can be interpreted as the angle between the vectors $z_2 - z_1$ and $z_4 - z_1$ as in Figure 14.1.2.

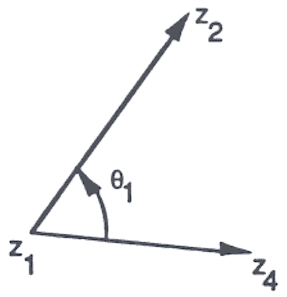

Figure 14.1.2

So we can write the cross ratio r of the 4 points, z_1, z_2, z_3, z_4, as follows:

$$r = \left(\frac{z_1 - z_2}{z_1 - z_4}\right)\left(\frac{z_3 - z_4}{z_3 - z_2}\right) = \left|\frac{z_1 - z_2}{z_1 - z_4}\right| e^{i\theta_1} \left|\frac{z_3 - z_4}{z_3 - z_2}\right| e^{i\theta_3}$$
$$= \left|\frac{z_1 - z_2}{z_1 - z_4}\right| \left|\frac{z_3 - z_4}{z_3 - z_2}\right| e^{i(\theta_1 + \theta_3)}$$

where θ_3 is the angle at z_3 in the quadrilateral determined by the 4 points z_1, z_2, z_3, z_4. See Figure 14.1.3.

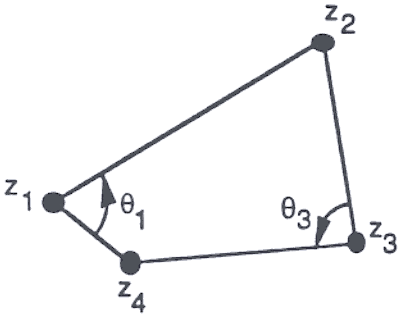

Figure 14.1.3

We conclude with a result that connects our complex geometry to circles.

Theorem 14.1.1: *The 4 distinct points z_1, z_2, z_3, z_4 in the complex projective line have a real cross ratio if and only if they all lie on a single circle or Euclidean line.*

Proof. From the discussion above z_1, z_2, z_3, z_4 have a real cross ratio if and only if $\theta_1 + \theta_3$ is an integral multiple of Π. But a result from Euclidean geometry says that $\theta_1 + \theta_3$ is an integral multiple of Π if and only if z_1, z_2, z_3, z_4 all lie on a single circle or Euclidean line.

Corollary 14.1.2: *The image of a circle or Euclidean line under a Moebius function f of the complex projective line is a circle or Euclidean line.*

Proof. Choose your favorite circle or Euclidean line and fix three distinct points z_1, z_2, z_3, on it. A fourth point z_4 lies on that circle or Euclidean line if and only if the cross ratio of z_1, z_2, z_3, z_4 is real. Similarly, the cross ratio of $f(z_1)$, $f(z_2)$, $f(z_3)$, $f(z_4)$ is real if and only if $f(z_4)$ lies on the circle or Euclidean line determined by $f(z_1)$, $f(z_2)$, $f(z_3)$. By Lemma 13.5.1 (the invariance of the cross ratio), the cross ratio of z_1, z_2, z_3, z_4 and $f(z_1)$, $f(z_2)$, $f(z_3)$, $f(z_4)$ are the same. So z_4 lies on the circle or Euclidean line through z_1, z_2, z_3 if and only if $f(z_4)$ lies on the circle or Euclidean line through $f(z_1)$, $f(z_2)$, $f(z_3)$.

14.2 Inversion

Recall from Chapter 13 that any Moebius function can be regarded as the composition of translations, multiplications by a constant, and taking the multiplicative inverse. Consider the Moebius function $f(z) = 1/z$. For the sake of tradition and for the sake of understanding the function more simply, we define a slightly different function. We call *inversion* the function defined by

$$\beta(z) = 1/\bar{z}.$$

Since complex conjugation is just a rigid reflection about the real axis, β takes circles and Euclidean lines to circles and Euclidean lines as well.

Note that $|z|^2\beta(x) = z\bar{z}/\bar{z} = z$. So z and $\beta(z)$ are on a ray from the origin. When $|z| = 1$, then $\beta(z) = z$. Inversion is like a "reflection" about a circle. Figure 14.2.1 shows the inversion of some lines and circles.

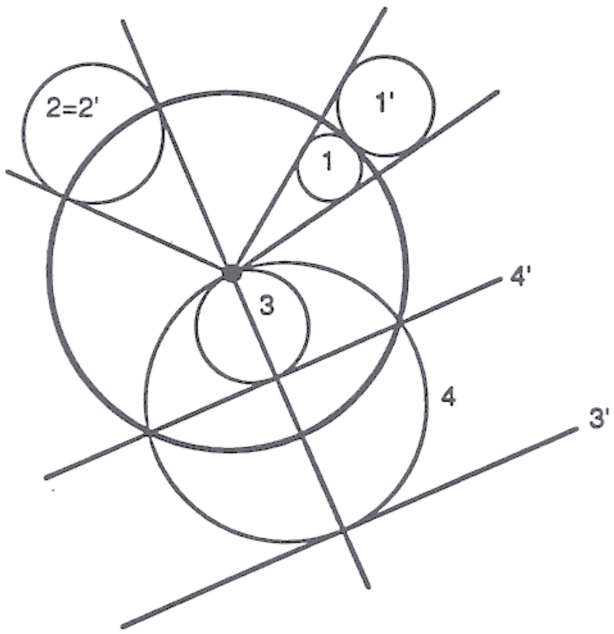

Figure 14.2.1

The following are some easy properties of inversion. Euclidean lines are thought of as circles through the single point at infinity.

1. For all z in the complex projective line, $\beta(\beta(z)) = z$, and $\beta(z) = z$ if and only if z is on the unit circle.
2. Rays and Euclidean lines though the origin are inverted into themselves.

3. Circles through the origin are inverted into Euclidean lines not through the origin, and vise-versa. For example, the circles 3 and 4 are inverted to the Euclidean lines $3'$ and $4'$, respectively, in Figure 14.2.1.
4. If two circles are tangent, or a circle and a Euclidean line are tangent, so are their inverted images. For example, in the Figure, Circle 1 is tangent to the unit circle and two rays through the origin. So its image, Circle $1'$, is also tangent to the same two rays and the unit circle since they are inverted to themselves.
5. A circle is inverted into itself if and only if it is either the unit circle or orthogonal to the unit circle. (Two circles are *orthogonal* if the tangent lines to one circle at the points of intersection go through the center of the other circle.) For example, in the Figure, Circle 2 is orthogonal to the unit circle and is inverted into itself. The points of intersection on the unit circle go into themselves as well as the two tangent rays, and this determines the circle uniquely.

14.3 Linkages

The first steam engines were used in England from 1712, and although they were inefficient, they rapidly came to be used widely. In 1765, James Watt, a mathematical instrument maker at the University of Glascow, invented a separate condenser improving the efficiency of the steam engine. But he needed a way of converting the back-and-forth reciprocal motion of the piston to the more convenient rotational motion of a flywheel.

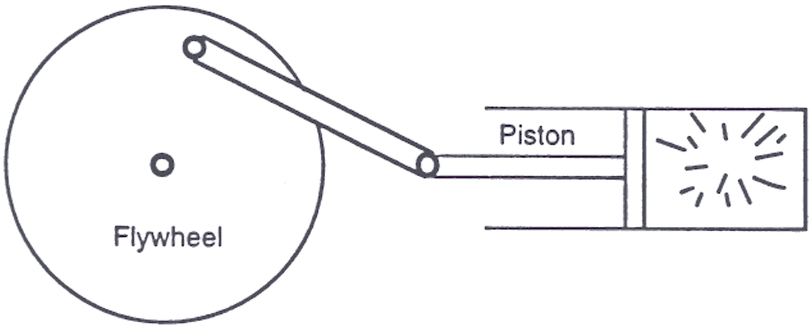

Figure 14.3.1

He did not find an exact solution, but he did find the following mechanism that was a solution good enough for the problem at hand.

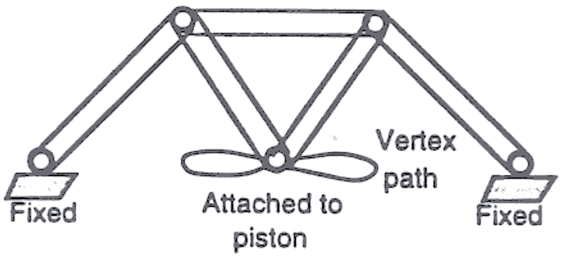

Figure 14.3.2

The point that was to be attached to the piston described a flattened figure eight path that was "almost" a straight line.

Mathematically the problem was to find a configuration of points in the plane, with some of the points fixed and some pairs of the points constrained to say a constant distance apart (they have rigid bars between them), such that some point follows a straight line path. Some well-known mathematicians, for example P. Tschebyscheff, worked on the problem for some time with no success. It was even suggested that the problem had no solution!

In 1864 a young Captain in the French Corps of Engineers by the name of Peaucellier announced that he had found a solution to the problem. A few years later a young Lithuanian, L. Lipkin, found essentially the same solution, which we describe below.

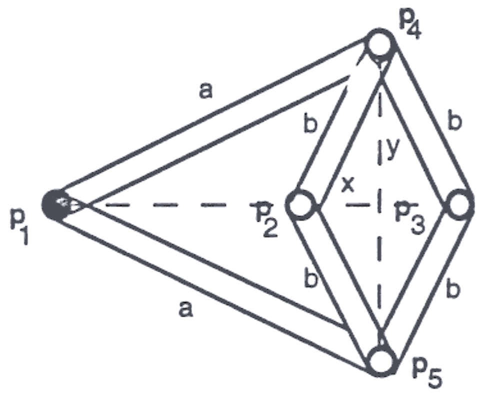

Figure 14.3.3

The idea is to find a mechanism that does inversion. In Figure 14.3.3 the black point is regarded as the center of the inversion, the origin in our description above. The indicated sides are equal. The points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are collinear because of the symmetry in teh lengths of the bars a and b. By the Theorem of Pythagoras applied to two right triangles,

$$a^2 = (|\mathbf{p}_2 \quad \mathbf{p}_1| + x)^2 + y^2 = |\mathbf{p}_2 - \mathbf{p}_1|^2 + 2x|\mathbf{p}_2 \quad \mathbf{p}_1| + x^2 + y^2$$
$$= |\mathbf{p}_2 - \mathbf{p}_1|^2 + 2x|\mathbf{p}_2 \quad \mathbf{p}_1| + b^2$$

Hence

$$a^2 - b^2 = |\mathbf{p}_2 - \mathbf{p}_1|^2 + 2x|\mathbf{p}_2 \quad \mathbf{p}_1|.$$

We calculate the product

$$|\mathbf{p}_2 \quad \mathbf{p}_1||\mathbf{p}_3 - \mathbf{p}_1 = |\mathbf{p}_2 - \mathbf{p}_1|(|\mathbf{p}_2 - \mathbf{p}_1| + 2x)$$
$$= |\mathbf{p}_2 - \mathbf{p}_1|^2 + 2x|\mathbf{p}_2 - \mathbf{p}_1| = a^2 + b^2$$

Thus if we arrange our units so that $a^2 - b^2 = 1$, then $\mathbf{p}_2$ and $\mathbf{p}_3$ will be inverted into each other with $\mathbf{p}_1$ as the center of inversion. (This is 0 in our description above.)

To finish the mechanism, we fix $\mathbf{p}_1$ and force $\mathbf{p}_2$ to lie on a circle that goes through $\mathbf{p}_1$. Thus $\mathbf{p}_3$ will lie on the inversion of the circle, which is a straight line. This is what was desired. Figure 14.3.4 shows the whole mechanism.

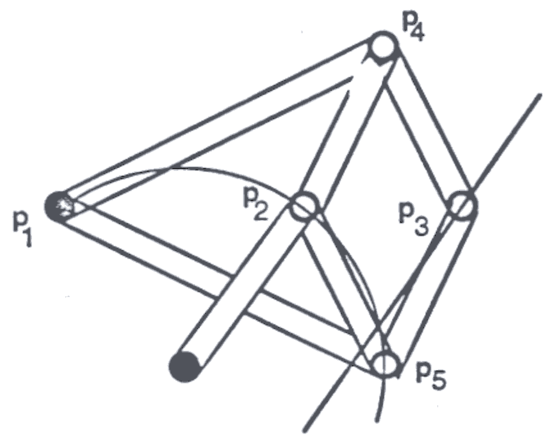

Figure 14.3.4

14.4 Stereographic projection

So far we have been working in the Euclidean plane, even though we have thought of it as a projective line. The principles of inversion still work in three-space, however, and we can take advantage of this to understand more of the geometry of both dimensions two and three.

Without regard to the complex structure inversion, in the Euclidean plane is just

$$\beta(\mathbf{p}) = \mathbf{p}/|\mathbf{p}|^2$$

In other words inversion simply takes a point along the ray from the center of inversion to a point whose distance from the center is the reciprocal of the distance of the original point from the center. We extend this definition to any Euclidean space.

One useful technique is to intersect the objects in three-space we are studying with appropriately chosen planes. This allows us to extend results from the plane to three-space. For example, what do we get when invert a sphere S through the origin, tangent to the unit sphere in three-space? See Figure 14.4.1.
Intersect S with a plane Π through 0 and the point of tangency with the unit sphere, which we have called the South Pole in the figure. The South Pole is fixed under the inversion β, and $\Pi \cap S$ is a circle through the origin 0. By the properties of inversion in a plane this circle is inverted into a line tangent to $\Pi \cap S$ in Π. So the line is tangent to S as well. These lines fill out a plane tangent to S at the South Pole. Thus $\beta(S)$ is the plane tangent to S at the South Pole.

In fact, this idea works for any sphere S in three-space. The line through the center of S intersects S st the endpoints $\mathbf{p}$ and $\mathbf{q}$ of a diameter of S. Any plane Π through this line intersects S in a circle, and all such circles have the same diameters $\mathbf{p}$ and $\mathbf{q}$. Again since β restricted to Π has the properties we listed in Section 14.2, $\beta(\Pi \cap S)$ is a circle with $\beta(\mathbf{p})$ and $\beta(\mathbf{q})$ as diameter, or a line perpendicular to the line through $\mathbf{p}$ and $\mathbf{q}$ if S contains 0. By rotating the plane Π around the line through $\mathbf{p}$ and $\mathbf{q}$ we see that $\beta(S)$ is a sphere, or a plane if S contains 0. See Figure 14.4.2.

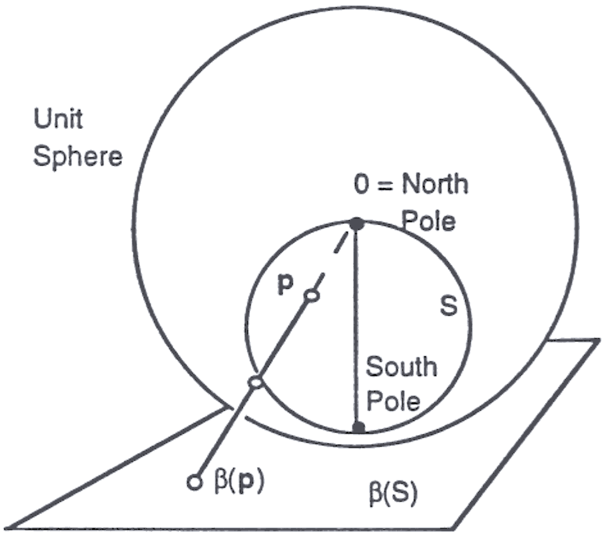

Figure 14.4.1

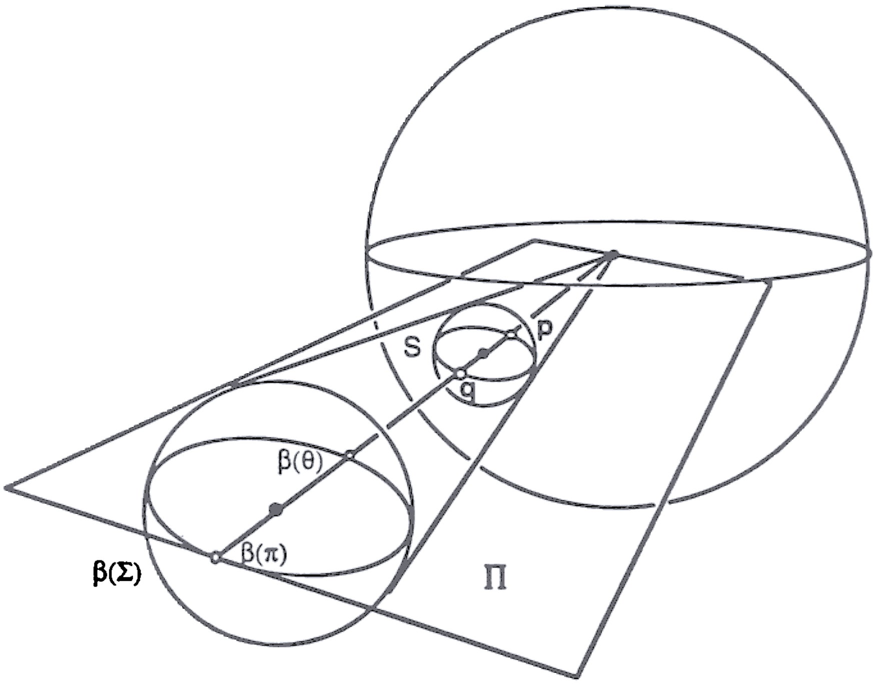

Figure 14.4.2

Now it is easy to see that the image under inversion of any circle, not just the one whose planes contain the center of inversion, is a circle or a line. This because the intersection of two spheres is a circle, and the the inversion of the intersection is the intersection of the inversion of each sphere, which is a sphere or plane. So the inversion of a circle is a circle or line as it is in the plane.

This is especially useful when the inversion is restricted to a sphere that contains the center of inversion. Suppose that a sphere S is tangent to a plane at a point we call the South Pole. Call the point antipodal to the South Pole, the North Pole. Inversion about the North Pole, with the unit length equal to the diameter of S, takes S into the tangent plane. This is called *stereographic projection.* Note that each point on S is projected onto a point in the tangent plane along a line through the North Pole. This is our usual notion of projection, but the domain is not a plane but a sphere. See Figure 14.4.1. We record the basic property of stereographic projection.

Theorem 14.4.1: *The image of a circle on the sphere under stereographic projection is either a circle in the plane or a line in the plane if the circle goes through the North Pole.*

This will be used as a further manifestation of our attitude that three dimensions helps greatly in understanding two dimensions.

Exercises:

1. Is there any circle C in the (Euclidean) plane such that the center of C is inverted into the center of the image of C? Why?
2. Which circles have their orientation reversed by inversion in the (Euclidean) plane? For example, circle 1 in Figure 14.2.1 has its orientation reversed as it is mapped into Circle 1′. Think of the orientation of a circle as the direction a bug goes, either clockwise or counterclockwise, as it goes around the circle.
3. Let $r : S \to S$ denote reflection about the equator on the sphere S used for stereographic projection. Show that $\beta r \beta^{-1}$ is inversion about the image of the equator, where β is stereographic projection. Find a similar description for the function that takes the multiplicative inverse of a complex number.
4. Consider the following configuration of four points z_1, z_2, z_3, z_4 in the complex projective line, where the white point is the center of the larger circle, and the lines through z_1 and z_2 are tangent to the smaller circle.

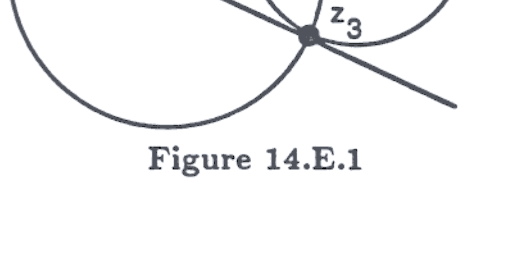

Figure 14.E.1

a. What is the cross ratio of z_1, z_2, z_3, z_4?
b. Show that the Figure above can be extended in the following way where z_1 and z_3 are collinear with the lower white point, and the lines from the lower white point to z_2 and z_4 are tangent to the smaller circle.

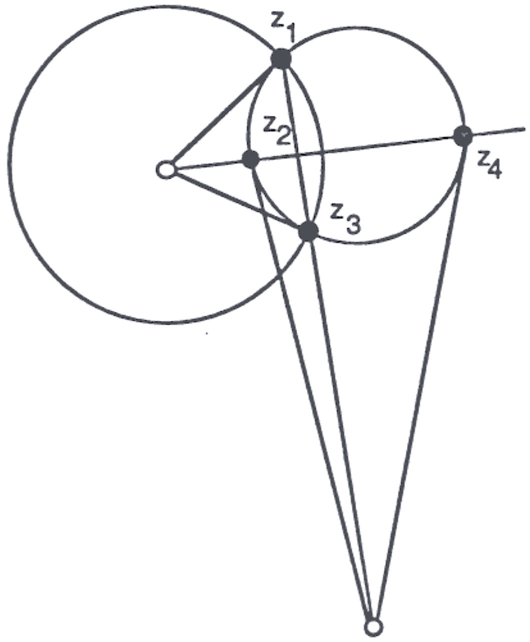

Figure 14.E.2

5. Suppose that the point **p** is inverted into the point $\mathbf{q} \neq \mathbf{p}$ with respect to the circle C.

a. Show that any circle through **p** and **q** is orthogonal to C. See property 5 in Section 14.2 for a definition of orthogonal.
b. Let **p** and **q** be two distinct points in the (Euclidean) plane. Consider the family of all circles and line through **p** and **q**. Show that there is another family of circles and line such that each element of the second family is orthogonal to each element of the first family, and every point in the plane, except **p** and **q**, is in one and only one element of the second family. See Figure 14.E.3. This is called a coaxal system in the old literature.

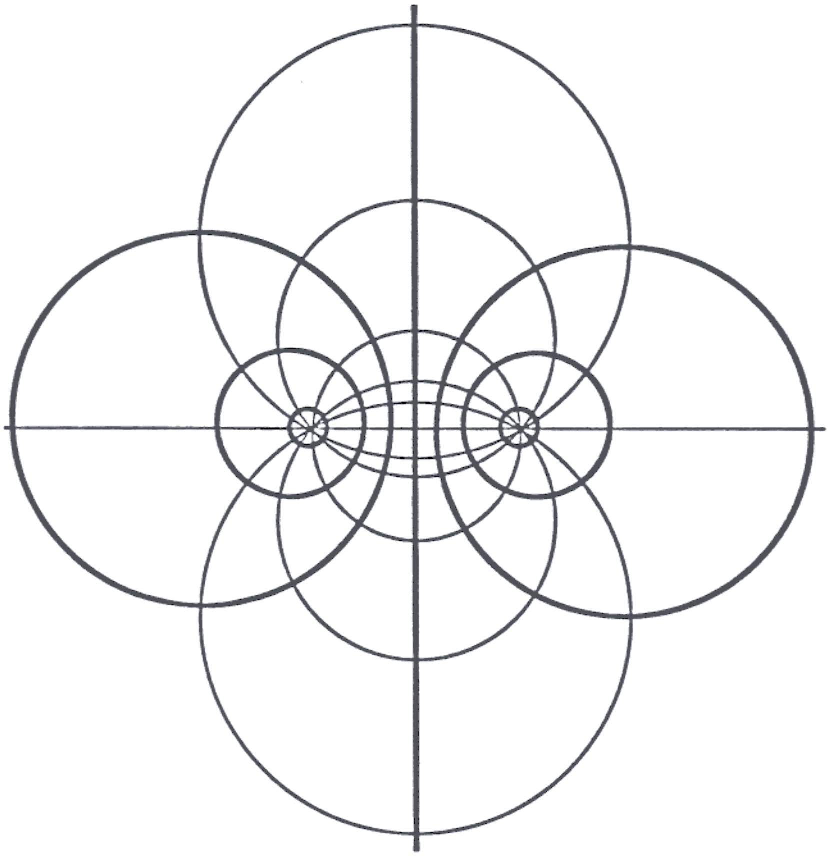

Figure 14.E.3