CLASSICAL GEOMETRIES

15. Hyperbolic geometry

So far we have talked mostly about the incidence structure of points, lines and circles. But geometry is concerned about the metric, the way things are measured. We also mentioned in the beginning of the course about Euclid's Fifth Postulate. Can it be proven from the the other Euclidean axiom's?

This brings up the subject of hyperbolic geometry. In the hyperbolic plane the parallel postulate is false. If a proof in Euclidean geometry could be found that proved the parallel postulate from the others, then the same proof could be applied to the hyperbolic plane to show that the parallel postulate is true, a contradiction. The existence of the hyperbolic plane shows that the Fifth postulate cannot be proven from the others. Assuming that Mathematics itself (or at least Euclidean geometry) is consistent, then there is no proof of the parallel postulate in Euclidean geometry. Our purpose in this chapter is to show that THE HYPERBOLIC PLANE EXISTS.

15.1 A quick history

In the first half of the nineteenth century people began to realize that that a geometry with the Fifth postulate denied *might* exist. N. I. Lobachevski and J. Bolyai essentially devoted their lives to the study of hyperbolic geometry. They wrote books about hyperbolic geometry, and showed that there there were many strange properties that held. If you assumed that one of these strange properties did not hold in the geometry, then the Fifth postulate could be proved from the others. But this just amounted to replacing one axiom with another equivalent one. These people simply assumed that there was such a non-Euclidean hyperbolic geometry. For all they knew, they could have been talking about the empty geometry, proving wonderful theorems about beautiful structures that do not exist. It has happened in other areas of Mathematics. Even the great C. F. Gauss only explored what might happen if this non-Euclidean geometry were really there. However, Gauss never actually published what he found, possibly out of fear of ridicule.

Nevertheless, by the middle of the nineteenth century the existence of the hyperbolic plane, even with its strange properties, came to be accepted, more or less. I think that is an example of the "smart people" argument, a variation of proof by intimidation. If many smart people have tried to find a solution to a problem and they do not succeed, then the problem must not have a solution. So in 1868, when E. Beltrami actually proved that one can construct the hyperbolic plane using standard mathematics and Euclidean geometry, perhaps it came as an anti-climax. From then on though, hyperbolic geometry was less of a mystery and part of the standard geometric repertoire. The ancient problem from Greek geometry "Can the Fifth postulate be proved from the others?" had been solved. The Fifth postulate cannot be proved.

We will present a construction for the hyperbolic plane that is a bit different in spirit from Beltrami's, and is in the spirit of Klein's philosophy, concentrating on the group of the geometry. This uses a seemingly unusual method, due to H. Minkowskii, that uses an analogue to an inner product that has non-zero vectors with a zero norm. Odd as that may seem, these ideas were fundamental to Einstein's special theory of relativity.

15.2 A little algebra

We will be working with special conics and quadratic curves and this brings up symmetric matrices. We will need some special information about these matrices.

A square matrix S is called *symmetric* if $S^t = S$, where $()^t$ denotes the transpose of a matrix.

Proposition 15.2.1: *Suppose that S is an n by n symmetric matrix over the real field such that for all vectors $\mathbf{p}$ in $\mathbf{R}^n$, $\mathbf{p}^t S\mathbf{p} = 0$. Then $S = 0$.*

For example, take the case when $n = 2$. Then

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

and let

$$\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then

$$\mathbf{p}^t S\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = ax^2 + 2bxy + cy^2$$

This is called a *quadratic form* in 2 variables. As an exercise you can prove that if this form is 0 on 3 vectors, every pair of which is independent, then the form is 0. In fact, we wil need a slightly stronger version of Proposition 15.2.1 where the form is 0 on some open subset of vectors in n-space.

15.3 The hyperbolic line and the unit circle

We need to study the lines in the hyperbolic plane, and in order to understand this we will work by analogy with the unit circle that is used in spherical geometry. We define them as follows:

The Unit Circle

$$\mathbf{S}^1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x^2 + y^2 = 1 \right\}$$

The Hyperbolic Line

$$\mathbf{H}^1 = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid x^2 - t^2 = -1, t > 0 \right\}$$

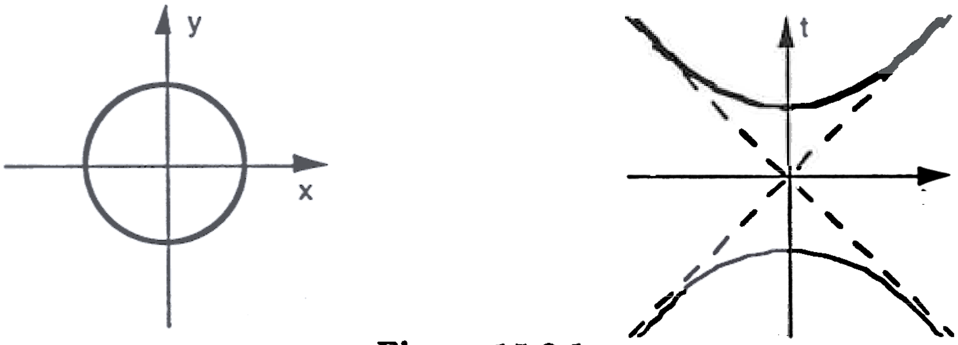

Figure 15.3.1

We rewrite these conditions in terms of matrices as follows:

The Circle

For every **p** and **q** in $\mathbf{R}^2$ define a "bi-linear form" by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}^t \mathbf{q},$$

where **p** and **q** are regarded as column vectors. So

$$\mathbf{S}^1 = \{\mathbf{p} \in \mathbf{R}^2 \mid \langle \mathbf{p}, \mathbf{p} \rangle = 1\}$$

where

$$\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The Hyperbola

For every **p** and **q** in $\mathbf{R}^2$ define a "bi-linear form" by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}^t D \mathbf{q},$$

where **p** and **q** are regarded as column vectors and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So

$$\mathbf{H}^1 = \{\mathbf{p} \in \mathbf{R}^2 \mid \langle \mathbf{p}, \mathbf{p} \rangle = -1\}$$

where

$$\begin{pmatrix} x \\ t \end{pmatrix}$$

There should be no confusion between the two bilinear forms since one is used only in the context of the circle and the other is used only in the context of the hyperbola. In the case of the circle, the bilinear form is the usual dot product.

One important difference between the two bilinear forms is that the form in the case of the hyperbola has vectors $\mathbf{p}$ such that $\langle \mathbf{p}, \mathbf{p} \rangle = 0$, but $\mathbf{p} \neq 0$. These are the vectors (called *isotropic* vectors) that lie along the asymptotes that are the dashed lines in Figure 15.3.1.

15.4 The group of transformations

Following the philosophy of Klein we define the group of transformations of the space, and use that to find the geometric properties. Each of our spaces in question, the circle and the hyperbola, are subspaces of the plane. We require that the group of transformations in question are a subgroup of the group of linear transformations. This is certainly the situation that we want for the circle, and we shall see that it gives us a useful group in the case of the hyperbola.

The Circle

We look for those 2 by 2 matrices A such that the image of S^1 is S^1 again. Let $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$. We look for those A such that

$$\mathbf{p} \in \mathbf{S}^1 \Leftrightarrow A\mathbf{p} \in \mathbf{S}^1 \Leftrightarrow (A\mathbf{p})^t A\mathbf{p} = \mathbf{p}^t A^t A\mathbf{p} = 1 = \mathbf{p}^t\mathbf{p}.$$

So

$$\mathbf{p}^t(A^tA - I)\mathbf{p} = 0,$$

where I is the identity matrix. The proof of Proposition 15.2.1 applies and we get

$$A^tA - I = 0.$$

So $A^tA = I$, which is the condition for being orthogonal.

The Hyperbola

We look for those 2 by 2 matrices A such that the image of $\mathbf{H}^1$ is $\mathbf{H}^1$ again. Let $\mathbf{p} = \begin{pmatrix} x \\ t \end{pmatrix}$. We look for those A such that

$$\mathbf{p} \in \mathbf{H}^1 \Leftrightarrow A\mathbf{p} \in \mathbf{H}^1 \Leftrightarrow (A\mathbf{p})^t DA\mathbf{p} = \mathbf{p}^t A^t DA\mathbf{p} = 1 = \mathbf{p}^t D\mathbf{p}.$$

So

$$\mathbf{p}^t(A^tDA - D)\mathbf{p} = 0,$$

where D is the matrix in 15.3. The proof of Proposition 15.2.1 applies and we get

$$A^tA - I = 0.$$

So $A^tDA = D$, which is similar to the condition for being orthogonal.

15.5 The metric: How to measure distances

If we have two pairs of points in the line, or in any space for that matter, how do we tell when they have the same distance apart? You might say that you just compute the distances. But how do you do that? Physically, you might use a ruler, but let us consider what that means. You must actually move the ruler from one pair of points to the other. But this motion must be in our group of "geometric" transformations. In the case of the circle and the hyperbolic line, we have already decided what that group of transformations is. The following principle states our point of view describing when two line segments have the same length.

Principle of Superposition: *Two line segments have the same length if and only if they can be superimposed by an element of the group of geometric transformations.*

In Section 15.4 we have described the group of geometric transformations by characterizing their matrices. We wish to make a further reduction. On a line or a circle there are two ways to superimpose two line segments. If we use directed line segments, say, and direct them all the same way, we can still require that they have the same length if and only if they can be superimposed by an element of the group. In fact, the elements of the groups that are defined in Section 15.4 form a subgroup where the determinate is 1. Call this restricted group the *positive transformations*. These are the transformations that can be regarded as the rotations of the circle or the translations of the hyperbolic line.

These positive transformations also have the property that there is a unique (positive) transformation that takes one point to another. Indeed we can modify or superposition principle by bringing all our directed line segments back to a fixed canonical position, which we call Paris, in order to compare lengths. For a while Paris did keep a fixed meter length that was used for comparison the world over. So there is a unique positive geometric transformation that takes a point to Paris. Alternatively, we can think of Paris being transformed to any given point θ by a positive transformation A_θ. We present A_θ explicitly and define Paris.

The Circle

Let Paris $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So A_θ on $\mathbf{S}^1$ is identified with

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

The Hyperbola

Let Paris $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$A_\theta = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

where the hyperbolic functions are defined by

$$\cosh\theta = (e^\theta + e^{-\theta})/2$$
$$\sinh\theta = (e^\theta - e^{-\theta})/2.$$

So A_θ on $\mathbf{H}^1$ is identified with

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sinh\theta \\ \cosh\theta \end{pmatrix}$$

We leave it as an easy exercise to show that these matrices A_θ satisfy the orthogonal and hyperbolic orthogonal conditions described in Section 15.4. Figure 15.5.1 shows the "ruler" in circle geometry as well as hyperbolic geometry.

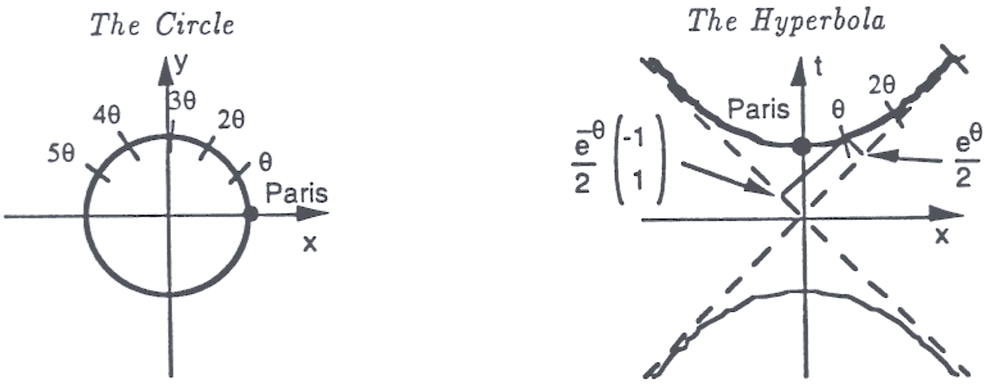

Figure 15.5.1

In terms of the ordinary Euclidean distance, multiples of θ grow exponentially on the hyperbolic line, whereas on the circle the multiples of θ appear at fixed intervals around the circle. Notice also that the isotropic directions

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

are also independent eigenvectors of A_θ, with eigenvalues $e^\theta/2$ and $e^{-\theta}/2$ respectively.

15.6 Computing length from coordinates

In Section 15.5 we saw *how* to compute distances by moving rulers around, but it would be helpful if we could assign a real number that was the distance between any pair of points in our space. Philosophically, we want this distance to "look like" the distance along a line. We describe this as follows:

The Principle of Juxtaposion: *The length of two intervals put end-to-end is the sum of the lengths of the two intervals.*

This principle does not strictly hold for the circle, which is one reason that spherical geometry or elliptic geometry does not even satisfy the first few axioms of Euclid. Nevertheless, this principle does hold "in the small", namely when the intervals are both sufficiently small. In hyperbolic geometry, this principle does hold and it tells us how to compute distances.

In either case, suppose that θ_1 and θ_2 are two points, both in $\mathbf{S}^1$ or both in $\mathbf{H}^1$. We can regard θ_1 and θ_2 as points on a line, but we must figure out how to add their lengths from Paris, say. One way is to arrange it so that $\theta_1 + \theta_2$ is that point where

$$A_{\theta_1+\theta_2} = A_{\theta_1} + A_{\theta_2}$$

It is easy to check that this equation is satisfied by the circular functions, sine and cosine, on the one hand, and the hyperbolic sine and hyperbolic cosine, on the other hand. So the distance between θ_1 and θ_2 is just $|\theta_1 - \theta_2|$ with the given parametrizations.

Suppose $\mathbf{p}$ and $\mathbf{q}$ are two points that we know by their Cartesian coordinates as we have described above. What is their distance? In order to make the calculation easier we will bring the interval between them back to Paris, and we will center it so that the midpoint is exactly at Paris. So then θ corresponds to $\mathbf{p}$ and $-\theta$ corresponds to $\mathbf{q}$. Let $d(\mathbf{p},\mathbf{q}) = 2\theta$ denote the distance between $\mathbf{p}$ and $\mathbf{q}$. Looking at the coordinates as below allows us to calculate $d(\mathbf{p},\mathbf{q})$.

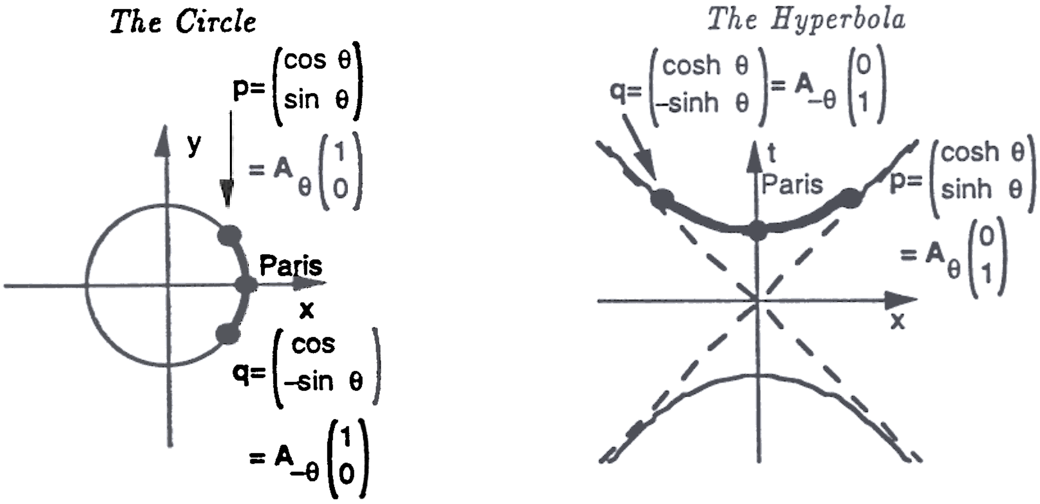

Figure 15.6.1

So the difference of **p** and **q** is

$$\mathbf{p}-\mathbf{q}=\begin{pmatrix}0\\ 2\sin\theta\end{pmatrix}$$

$$\langle \mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle = 4\sin^2\theta.$$

$$d(\mathbf{p},\mathbf{q}) = 2\theta$$

$$= 2\sin^{-1}\left[\frac{1}{2}\sqrt{\langle \mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle}\right].$$

So the difference of **p** and **q** is

$$\mathbf{p}-\mathbf{q}=\begin{pmatrix}2\sinh\theta\\ 0\end{pmatrix}$$

Thus

$$\langle \mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle = 4\sinh^2\theta.$$

$$d(\mathbf{p},\mathbf{q}) = 2\theta$$

$$= 2\sinh^{-1}\left[\frac{1}{2}\sqrt{\langle \mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle}\right].$$

Notice that we have written the distance in terms of the bilinear form $\langle , \rangle$. If we replace **p** and **q** by $A\mathbf{p}$ and $A\mathbf{q}$ respectively, where A is an orthogonal or hyperbolic orthogonal matrix, then we compute that the bilinear form is invariant.

Circle Case

$$\begin{aligned}\langle \mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle &= (\mathbf{p}-\mathbf{q})^t(\mathbf{p}-\mathbf{q})\\ &= (\mathbf{p}-\mathbf{q})^t A^t A(\mathbf{p}-\mathbf{q})\\ &= [A(\mathbf{p}-\mathbf{q})]^t[A(\mathbf{p}-\mathbf{q})]\\ &= \langle A\mathbf{p}-A\mathbf{q}, A\mathbf{p}-A\mathbf{q}\rangle.\end{aligned}$$

Hyperbolic Case

$$\begin{aligned}\langle \mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle &= (\mathbf{p}-\mathbf{q})^t D(\mathbf{p}-\mathbf{q})\\ &= (\mathbf{p}-\mathbf{q})^t A^t DA(\mathbf{p}-\mathbf{q})\\ &= [A(\mathbf{p}-\mathbf{q})]^t D[A(\mathbf{p}-\mathbf{q})]\\ &= \langle A\mathbf{p}-A\mathbf{q}, A\mathbf{p}-A\mathbf{q}\rangle.\end{aligned}$$

Thus the formula (*) in terms of the bilinear form is valid for any **p** and **q**. This is an explicit function of the coordinates.

15.7 An extra dimension

We are finally ready for the hyperfolic plane itself. The proper analogy is the 2-sphere. We define both the sphere and the hyperbolic plane to show their similarities.

The Sphere

Define

$$\mathbf{S}^2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x^2 + y^2 + z^2 = 1 \right\}$$

$$= \{\mathbf{p} \in \mathbf{R}^3 \quad \langle \mathbf{p}, \mathbf{p} \rangle = 1\},$$

where

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}^t \mathbf{q}.$$

The orthogonal matrices are those matrices A such that

$$A^t A = I.$$

$$d(\mathbf{p}, \mathbf{q}) = 2 \sin^{-1} \left[\frac{1}{2} \sqrt{\langle \mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} \rangle} \right].$$

The Hyperbolic Plane

Define

$$\mathbf{H}^2 = \left\{ \begin{pmatrix} [illegible] \end{pmatrix} \right\}$$

$$= \{\mathbf{p} \in \mathbf{R}^3 \quad \langle \mathbf{p}, \mathbf{p} \rangle = -1\},$$

where

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}^t D \mathbf{q},$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The hyperbolic orthogonal matrices are those matrices A such that

$$A^t D A = D$$

The distance between $\mathbf{p}$ and $\mathbf{q}$ is

$$d(\mathbf{p}, \mathbf{q}) = 2 \sinh^{-1} \left[\frac{1}{2} \sqrt{\langle \mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} \rangle} \right]$$

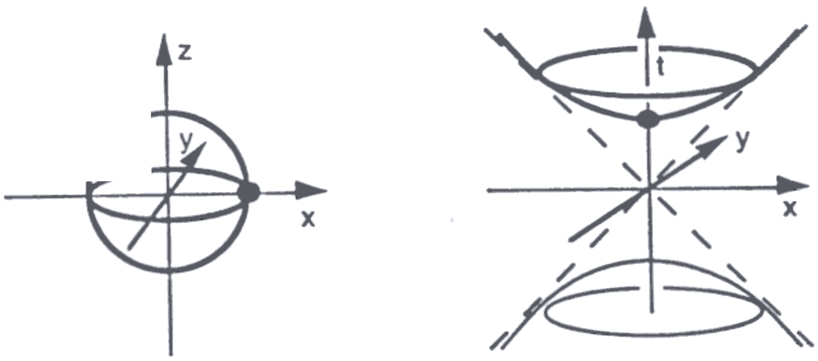

Figure 15.7.1

This gives the metric for the hyperbolic plane as well as for the two-sphere. It is also easy to see what the lines in these geometries are. Recall that for the two-sphere, a line was defined to be the intersection of a plane through the origin with the two-sphere. In the case of the hyperbolic plane practically the same definition works. Namely, a line is the intersection of a plane through the origin with $\mathbf{H}^2$, defined as the hyperbolist of revolution as defined above.

In both of these cases it is easy to see that any line can be transformed by an element of the group of geometric transformations to the circle or line that we defined previously in earlier sections. In the case of $\mathbf{S}^2$, rotations about each of the x-axis, y-axis, and z-axis generate the group of positive transformations. In the case of $\mathbf{H}^2$ the following matrices generate the hyperbolic transformations:

$$\begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix}, \begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In fact we now have the complete definition of the hyperbolic plane that is the notorious competitor of the Euclidean plane for satisfying the axioms of Euclidean geometry, except for the Fifth postulate. We have defined distances, but it follows (and is straightforward to do) that angles can also be defined for the hyperbolic plane. Bring everything to Paris, and measure it there.

15.8 The Klein-Beltrami model

An important feature of the hyperbolic plane is the property that it does not satisfy Euclid's Fifth postulate. How do we see this? simply look at $\mathbf{H}^2$. A good vantage point is the origin. Project $\mathbf{H}^2$ from the origin into the plane $t = 1$. Define this central projection $\pi : \mathbf{H}^2 \to \mathbf{R}^2$ by

$$\pi \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} x/t \\ y/t \\ 1 \end{pmatrix}$$

It is easy to see that π is a one-to-one function and that the image of $\mathbf{H}^2$ is the interior of the unit disk in the plane $t = 1$. See Figure 15.8.1.

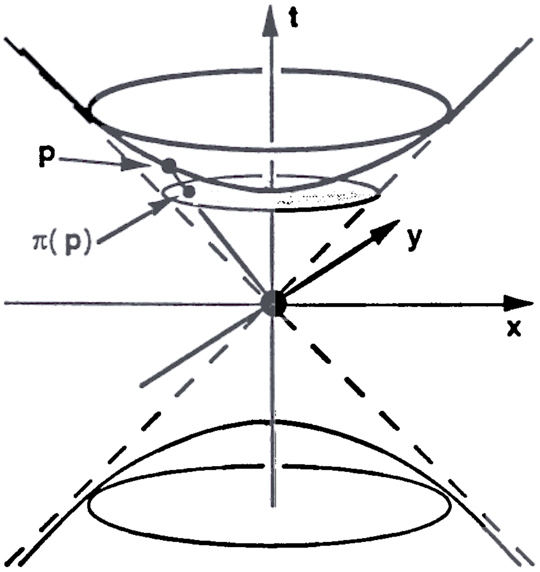

Figure 15.8.1

Since a line in $\mathbf{H}^2$ is the intersection of a plane through the origin with $\mathbf{H}^2$ itself, the projection of a line is an open line segment in the unit disk in the plane $t = 1$. It is easy to see that given any point $\mathbf{p}$ and a line $\mathbf{L}$ not containing that point, then there are many lines through $\mathbf{p}$ not intersecting $\mathbf{L}$. See Figure 15.8.2.

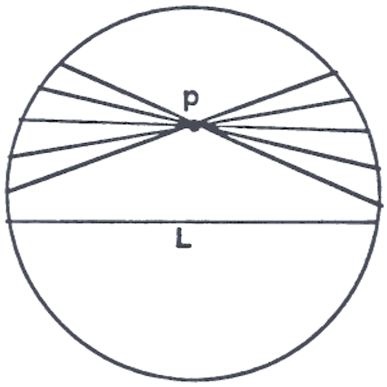

Figure 15.8.2

In this model we see that the hyperbolic distance between two points $\mathbf{p}$ and $\mathbf{q}$ is

$$d(\mathbf{p},\mathbf{q}) = 2\sinh^{-1}\left[\frac{1}{2}\sqrt{\langle \pi^{-1}\mathbf{p} - \pi^{-1}\mathbf{q}, \pi^{-1}\mathbf{p} - \pi^{-1}\mathbf{q}\rangle}\right].$$

We could use the above formula to define distances in this Klein model directly if we wished. Beltrami used a different approach that is common in differential geometry. He defined a real valued function at each point in the unit disk, and then the distance along a curve is obtained by integrating that function along the curve. The distance between two points is the shortest hyperbolic length of a curve between the two points.

15.9 The Poincaré model

The Klein model of the hyperbolic plane has the property that hyperbolic lines are "straight", but one might wish for other properties. We describe here a model, due to H. Poincaré, that is yet another equivalent description of the hyperbolic plane, but it has the pleasing property that hyperbolic circles are Euclidean circles.

For the Klein-Beltrami model we used central projection of $\mathbf{H}^2$ into the plane $t = 1$. Recall that for the sphere $\mathbf{S}^2$ stereographic projection took circles to circles. We can apply the same idea for $\mathbf{H}^2$ in our Minkowskii geometry. A natural place to choose the projection point is the antipode of Paris, namely $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = S$. See Figure 15.9.1.

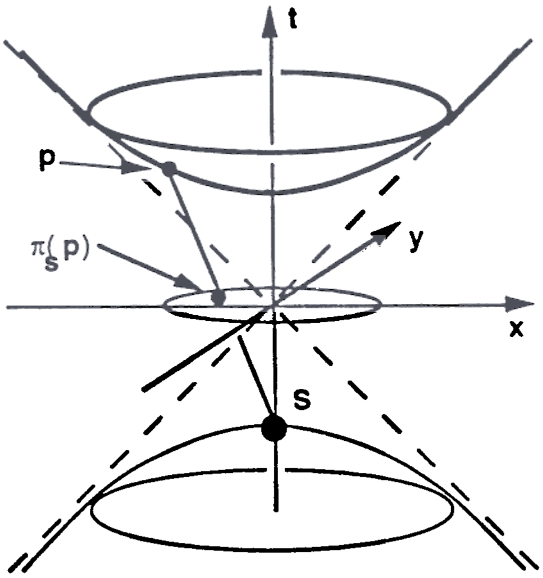

Figure 15.9.1

We define $\pi_s : \mathbf{H}^2 \to \mathbf{R}^2$ by projection from S into the plane $t = 0$. The image of $\mathbf{H}^2$ under π_s is the unit disk again, but in the plane $t = 0$. But what is the image of a hyperbolic line? It turns out that they are circles in the unit disk that are orthogonal to the boundary of the unit disk or they are diameters in the unit disk. So Figure 15.8.2 then is transformed into Figure 15.9.2.

An important property of the Poincaré model is that the angle between the circles that are the image of two lines is the same as the angle between the lines in the hyperbolic plane $\mathbf{H}^2$ itself. We say that the model is *conformal.* (It is also true that stereographic projection from the sphere $\mathbf{S}^2$ into the plane is conformal. The whole plane is the conformal image of the sphere $\mathbf{S}^2$ minus one point.)

We show some tilings M. C. Escher based on this conformal property at the end of the Chapter.

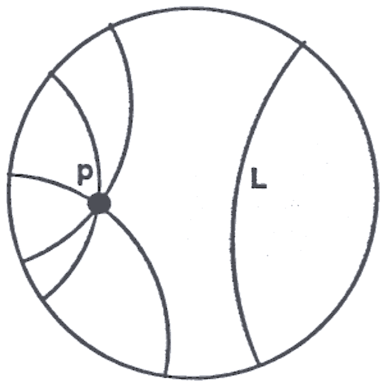

Figure 15.9.2

Exercises:

1. Show that if a real bilinear form in two variables is 0 on three independent vectors, then is is the 0 form.
2. In any dimension, show that if a real bilinear form is 0 on an open set of vectors, then it is the 0 form.
3. Show that the circular functions, sine and cosine, satisfy the matrix product rule in Section 15.5. Do the same for the hyperbolic functions.
4. Suppose that two lines in $\mathbf{H}^2$ are given by

$$b_1 \quad c_1\,) D \begin{pmatrix} x \\ y \\ t \end{pmatrix} = a_1 x + b_1 y - c_1 t = 0$$

$$b_2 \quad c_2\,) D \begin{pmatrix} x \\ y \\ t \end{pmatrix} = a_2 x + b_2 y - c_2 t = 0.$$

   Find the angle between them.
5. In Section 15.8 find an explicit representation of the inverse projection map π^{-1} that takes the unit disk onto $\mathbf{H}^2$.
6. Consider the unit sphere tangent to a plane at its South Pole as in Figure 15.E.1.

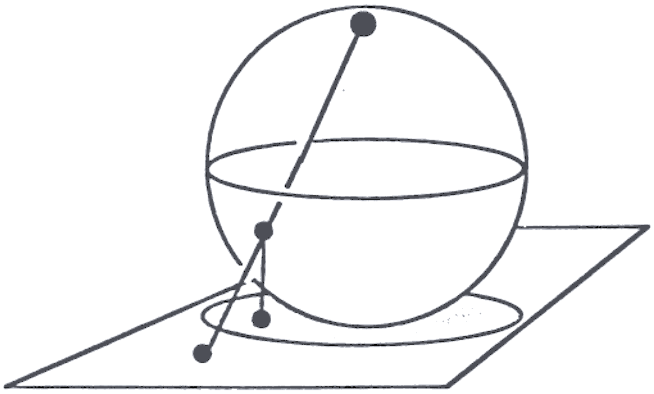

Figure 15.E.1

Projection parallel to the North-South axis of the sphere takes the unit disk centered at the South Pole to the Southern Hemisphere. Stereographic projection from the North Pole then takes the Southern Hemisphere onto the disk of radius 2 centered at the South Pole, back in the plane. Show that the composition of these two projections takes the Klein-Beltrami model of the hyperbolic plane onto the Poincaré model of the hyperbolic plane.

Show that the above composition of projections is also the same as the composition of projections using central projection to $\mathbf{H}^2$ and then the hyperbolic stereographic projection back to the plane followed by simple multiplication by 2 from the South Pole.

7. Show that the circumference of the disk in $\mathbf{H}^2$ of radius θ is $2\pi \sinh \theta$. This circumference is the set of points of distance θ from a fixed point. (Hint: Choose the center point to be Paris, and think about what kind of circle you have in $\mathbf{H}^2$.) What is the circumference of the disk of radius θ in $\mathbf{S}^2$?
8. Use Problem 7 to show that the area of the disk of radius θ in $\mathbf{H}^2$ is $2\pi(\cosh \theta - 1)$. You may use calculus. Think of a geometric way of finding the circumference of a disk by taking a derivative. What is the area of a disk of radius θ in $\mathbf{S}^2$?
9. Use Problem 8 to calculate the area in $\mathbf{H}^2$ of the annular region between the circle of radius θ and the circle of radius $\theta + 1$. What is the limit as θ goes to infinity of the ratio of this annular area and the total area of the disk of radius $\theta + 1$? Where is most of the area in a hyperbolic disk?