Math 6520 Homework due Wednesday 5 September 2018

Let E and F be finite-dimensional real vector spaces. Choose linear isomorphisms $\phi\colon E \to \mathbf{R}^n$ and $\psi\colon F \to \mathbf{R}^m$. Define a subset U of E to be *open* if $\phi(U)$ is open in $\mathbf{R}^n$. Let $U \subseteq E$ and $V \subseteq F$ be open and let $f\colon U \to V$ be a map. Define f to be *smooth* if $\psi \circ f \circ \phi^{-1}$ is smooth (C^∞). These notions of openness and smoothness do not depend on the choice of ϕ and ψ. (You can check this for yourself. No need to turn in a proof.)

It is useful in practice to have slightly more flexible notions of a chart and an atlas than the ones introduced in class. Let M be an arbitrary set. A *chart* on M is a triple (U, ϕ, E), where U is a subset of M, E is a finite-dimensional real vector space, and ϕ is a bijection from U onto an open subset of E. An *atlas* on M is a collection of charts $\mathscr{A} = \{ (U_i, \phi_i, E_i) \mid i \in I \}$ with the following properties: $\bigcup_{i\in I} U_i = M$ and for all $i, j \in I$ the set $\phi_i(U_i \cap U_j)$ is open in E_i, the set $\phi_j(U_i \cap U_j)$ is open in E_j, and the transition map

$$\phi_j \circ \phi_i^{-1}\colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is smooth.

1. Let $\mathscr{A}$ be an atlas on a set M in the above sense.
   (a) Show that there is a unique topology on M making each U_i an open subset and each ϕ_i a homeomorphism from U_i onto its image.
   (b) For each $i \in I$ choose a linear isomorphism $\lambda_i\colon E_i \to \mathbf{R}^{n_i}$. Let $\phi_i' = \phi_i \circ \lambda_i$. Prove that $\mathscr{A}' = \{ (U_i, \phi_i') \mid i \in I \}$ is an atlas on M in the usual sense (as defined in class).

If E and F are finite-dimensional vector spaces, then $\mathrm{Hom}(E, F)$ is by definition the set of all linear maps from E to F. This is a finite-dimensional vector space in its own right.

2. The *Grassmannian of k-planes* in $\mathbf{R}^n$ is the set $\mathrm{Gr}(n, k)$ of all k-dimensional linear subspaces of $\mathbf{R}^n$. For $E \in \mathrm{Gr}(n, k)$ let $E^\perp$ denote the orthocomplement of E relative to the standard inner product. Define $U_E = \{ F \in \mathrm{Gr}(n, k) : F \cap E^\perp = \{0\} \}$.
   (a) Show that every $F \in U_E$ is the graph of a unique linear map $T_F\colon E \to E^\perp$.
   (b) Define $\phi_E\colon U_E \to \mathrm{Hom}(E, E^\perp)$ by $\phi_E(F) = T_F$. Show that the collection $\{ (U_E, \phi_E, \mathrm{Hom}(E, E^\perp)) : E \in \mathrm{Gr}(n, k) \}$ constitutes an atlas on $\mathrm{Gr}(n, k)$ in the sense of Problem 1. (How to write the transition maps for the Grassmannian? Given two k-planes E_1 and E_2 in $\mathbf{R}^n$, let us denote the corresponding charts on $\mathrm{Gr}(n, k)$ by (U_1, ϕ_1) and (U_2, ϕ_2). The transition map $\phi_{21} = \phi_2 \circ \phi_1^{-1}$ has domain a subset of $\mathrm{Hom}(E_1, E_1^\perp)$ and target a subset of $\mathrm{Hom}(E_2, E_2^\perp)$. Let π_2 denote the orthogonal projection $\mathbf{R}^n \to E_2$ and $\pi_2^\perp$ the orthogonal projection $\mathbf{R}^n \to E_2^\perp$. Show that for a linear map $T\colon E_1 \to E_1^\perp$ we have

   $$\phi_{21}(T) = \pi_2^\perp \circ (\mathrm{id}_{E_1} \times T) \circ \big(\pi_2 \circ (\mathrm{id}_{E_1} \times T)\big)^{-1}.$$

   Here $\mathrm{id}_{E_1} \times T$ denotes the linear map $E_1 \to \mathbf{R}^n$ which sends x to $(x, T(x)) \in E_1 \times E_1^\perp = \mathbf{R}^n$.)
   (c) What is the dimension of $\mathrm{Gr}(n, k)$?

3. This is a calculus review problem.

(a) Let $g\colon [0,1] \to \mathbf{R}$ be a C^k function. Prove Taylor's formula with integral remainder term:

$$g(1) = \sum_{j=0}^{k-1} \frac{1}{j!} g^{(j)}(0) + \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} g^{(k)}(t)\, dt.$$

(Use induction on k and integration by parts.)

(b) For the remainder of this problem let U be an open subset of $\mathbf{R}^n$ and let $f \in C^k(U)$. Let $x \in U$ and let $h \in \mathbf{R}^n$ be any vector such that $x + th \in U$ for all $t \in [0,1]$. Prove the multivariable Taylor formula:

$$f(x+h) = \sum_{|\alpha|<k} \partial^\alpha f(x) \frac{h^\alpha}{\alpha!} + k \sum_{|\alpha|=k} \left(\int_0^1 (1-t)^{k-1} \partial^\alpha f(x+th)\, dt \right) \frac{h^\alpha}{\alpha!}.$$

(Put $g(t) = f(x+th)$, apply (a) and use the chain rule. See below for the notation.)

(c) Now suppose U is *star-shaped* with respect to the origin, i.e. $0 \in U$ and for all $x \in U$ the closed line segment from 0 to x is contained in U. Find functions $f_i \in C^{k-1}(U)$ and $f_{ij} \in C^{k-2}(U)$ given by explicit formulas such that

$$\begin{aligned} f(x) &= f(0) + \sum_{i=1}^n x_i f_i(x) \\ &= f(0) + \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^n x_i x_j f_{ij}(x) \end{aligned}$$

for all $x \in U$.

Explanation of the notation in Problem 3: a *multi-index* is an n-tuple of natural numbers

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{N}^n.$$

The *order* of α is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The *factorial* of α is $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$. The *multinomial coefficient* for α is $\binom{k}{\alpha} = k!/\alpha!$, where $k = |\alpha|$. The multinomial coefficient is equal to the number of ways of labelling k things with the numbers 1, 2,..., n, such that the label i is repeated α_i times. If h is a vector in $\mathbf{R}^n$, we denote the monomial $h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}$ by h^α. The name "multinomial coefficient" comes from the multinomial theorem $(h_1 + h_2 + \cdots + h_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} h^\alpha$. If f is a sufficiently differentiable function on an open subset of $\mathbf{R}^n$, we denote the k-fold partial derivative

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$$

by $\partial^\alpha f$.