Math 6520 Homework due Wednesday 19 September 2018

Upper half-space $\mathbf{H}^n$ is the collection of vectors $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ satisfying $x_n \geq 0$. Its *boundary* $\partial\mathbf{H}^n$ is the collection of vectors $x \in \mathbf{R}^n$ satisfying $x_n = 0$. We identify $\partial\mathbf{H}^n$ with $\mathbf{R}^{n-1}$ via the embedding $x \mapsto (x, 0)$ of $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$. In the next problem we will mean by a *chart* on a topological space M a pair (U, ϕ) with U open in M and $\phi\colon U \to \mathbf{H}^n$ a map which is a homeomorphism onto an open subset of $\mathbf{H}^n$ (in the subspace topology). As usual, a pair of such charts (U, ϕ) and (V, ψ) is *compatible* if the transition map $\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.[1] An *atlas* on M is a collection of compatible charts whose domains cover M, and all of which take values in $\mathbf{H}^n$ for the same value of n. A *maximal* such atlas $\mathscr{A}$ defines on M the structure of an *n-manifold with boundary*. Let us fix an n-manifold with boundary $(M, \mathscr{A})$. A point $x \in M$ is a *boundary point* if $\phi(x) \in \partial\mathbf{H}^n$ for some chart $(U, \phi) \in \mathscr{A}$ at x. The set of boundary points is called the *boundary* and denoted by ∂M. The tangent bundle of M is defined as for an ordinary manifold, namely as a quotient $TM = M'/{\sim}$, where

$$M' = \{\, (U, \phi, x, h) \mid (U, \phi) \in \mathscr{A},\ x \in U,\ h \in \mathbf{R}^n \,\}$$

and $(U, \phi, x, h) \sim (V, \psi, y, k)$ if $y = x$ and $k = D(\psi \circ \phi^{-1})(\phi(x))h$. A tangent vector $[U, \phi, x, h]$ at a boundary point $x \in \partial M$ is *tangent to the boundary* if $h_n = 0$. The tangent vector *points inward* (resp. *outward*) if $h_n \geq 0$ (resp. < 0).

1. Prove the following assertions.

(a) A point $x \in M$ is a boundary point if and only if $\phi(x) \in \partial\mathbf{H}^n$ for *all* charts $(U, \phi) \in \mathscr{A}$ at x.

(b) The boundary ∂M is an $n-1$-manifold.

(c) Let $x \in \partial M$ and $v = [U, \phi, x, h] \in T_x M$. The notions of "being tangent", "pointing inward", "pointing outward" are well-defined, i.e. independent of the representative (U, ϕ, x, h) of v. The set $T_x M$ is a vector space and $T_x \partial M$ is the subspace consisting of all vectors tangent to the boundary.

Let M be a manifold (without boundary) and $f\colon M \to \mathbf{R}^r$ a smooth map. Recall from class that $df\colon TM \to \mathbf{R}^r$ is defined as the composition $df = \mathrm{pr}_2 \circ Tf$ of the tangent map $Tf\colon TM \to T\mathbf{R}^r$ with the projection onto the second factor $\mathrm{pr}_2\colon T\mathbf{R}^r = \mathbf{R}^r \times \mathbf{R}^r \to \mathbf{R}^r$. Written in coordinates df is just the total derivative of f. That is to say, with respect to a chart (U, ϕ) of M, $df \circ T\phi^{-1}\colon \phi(U) \times \mathbf{R}^n \to \mathbf{R}^r$ is given by

$$df \circ T\phi^{-1}(x, v) = D(f \circ \phi^{-1})(x)v.$$

For $x \in M$ we write $d_x f$ (or sometimes df_x) for the linear map $\mathrm{pr}_2 \circ T_x f\colon T_x M \to \mathbf{R}^r$. If $r = 1$, $d_x f$ is a cotangent vector, i.e. an element of the dual vector space $T_x^* M = (T_x M)^*$.

2. Let M be a manifold and let $x \in M$. Let $[f] = [f]_x$ denote the germ at x of a smooth function f defined in a neighbourhood of x. Let $C^\infty_{M,x}$ be the algebra of germs at x. Define $\mathfrak{m}_x$ to be the set of all germs $[f]$ at x with the property that $f(x) = 0$, i.e. $\mathfrak{m}_x$ is the kernel of the evaluation map $\mathrm{ev}_x\colon C^\infty_{M,x} \to \mathbf{R}$. Prove the following assertions.

(a) The set $\mathfrak{m}_x$ is a maximal ideal of the algebra $C^\infty_{M,x}$, and the *residue field*, i.e. the quotient $C^\infty_{M,x}/\mathfrak{m}_x$, is isomorphic to $\mathbf{R}$.

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[1] The domain and range of the transition map may not be open subsets of $\mathbf{R}^n$. If A is any subset of $\mathbf{R}^n$, a map $f\colon A \to \mathbf{R}^m$ is *smooth* if for every $x \in A$ there exist an open neighbourhood U and a smooth map $\tilde{f}\colon U \to \mathbf{R}^m$ such that $\tilde{f}|_{U \cap A} = f|_{U \cap A}$. If B is any subset of $\mathbf{R}^m$, a map $f\colon A \to B$ is a *diffeomorphism* if f is smooth viewed as a map $A \to \mathbf{R}^m$, and has a smooth two-sided inverse $f^{-1}\colon B \to A$.

(b) The algebra $C^\infty_{M,x}$ is *local* in the sense that it has a unique maximal ideal.

(c) For each germ $[f] \in \mathfrak{m}_x$ define a covector $\eth([f]) \in T^*_xM$ by $\eth([f]) = d_xf$. Then $\eth([f])$ is well-defined and the map $\eth\colon \mathfrak{m}_x \to T^*_xM$ which sends $[f]$ to $\eth([f])$ is surjective.

(d) The kernel of $\eth$ is the *square* $\mathfrak{m}_x^2$ of the ideal $\mathfrak{m}_x$, i.e. the ideal generated by all products $[f][g]$ with $[f]$ and $[g]$ in $\mathfrak{m}_x$. The map $\eth$ induces an $\mathbf{R}$-linear isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong T^*_xM$.

The result of (d) offers yet another route to the tangent space, namely we can first define the cotangent space T^*_xM as the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, and then the tangent space T_xM as the dual of T^*_xM! (This definition is favoured by algebraic geometers because it works better for singular varieties.) The result generalizes as follows: let $\mathbf{F}$ be a field and A a local $\mathbf{F}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{F}$. Then $\mathrm{Der}_{\mathbf{F}}(A,\mathbf{F}) \cong \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}/\mathfrak{m}^2,\mathbf{F})$.

Let V be a real $n+1$-dimensional vector space. Let $\mathbf{P}V$ be the projectivization of V, i.e. the space of lines (1-dimensional linear subspaces) of V. Recall that, as a topological space, $\mathbf{P}V$ is the quotient of $V \setminus \{0\}$ by the equivalence relation $u \sim v \iff u = \lambda v$ for some real $\lambda \neq 0$. We write $[v]$ for the line determined by a nonzero vector $v \in V$. Recall also that $\mathbf{P}V$ is an n-manifold with an atlas consisting of charts (U,ϕ) defined as follows. For each ordered basis $\mathscr{B} = (e_0, e_1, \dots, e_n)$ of V, let $[u_0, u_1, \dots, u_n]$ be the homogeneous coordinates on $\mathbf{P}V$ determined by $\mathscr{B}$. (I.e. for each nonzero vector $(u_0, u_1, \dots, u_n) \in \mathbf{R}^{n+1}$, $[u_0, u_1, \dots, u_n]$ is the line spanned by the vector $u_0e_0 + u_1e_1 + \cdots + u_ne_n$.) Let $U = U_{\mathscr{B}}$ be the set of lines $[u_0, u_1, \dots, u_n]$ with $u_0 \neq 0$. Define the coordinate map $\phi = \phi_{\mathscr{B}}\colon U_{\mathscr{B}} \to \mathbf{R}^n$ by

$$\phi([u_0, u_1, \dots, u_n]) = \frac{1}{u_0}(u_1, u_2, \dots, u_n).$$

The *incidence relation* $\tilde{V}$ is the set of all pairs (l, v) in $\mathbf{P}V \times V$ such that v is contained in l. Define $\beta\colon \tilde{V} \to V$ by $\beta(l,v) = v$. The *exceptional fibre* of β is $E = \beta^{-1}(0)$.

3. Prove the following assertions.

(a) $\tilde{V}$ is a closed submanifold of $\mathbf{P}V \times V$ of codimension n. (This can be done in several ways. For instance, note that it is enough to show $\tilde{V} \cap (U \times V)$ is a codimension n-submanifold for every open U as above. Using coordinates $(u_1, u_2, \dots, u_n, v_0, v_1, \dots, v_n)$ on the open subset $U \times V \cong \mathbf{R}^n \times \mathbf{R}^{n+1}$, show that $\tilde{V}$ is given by a system of equations $v_1 = v_0u_1$, $v_2 = v_0u_2$,$\dots$)

(b) β is a smooth map and, for every nonzero $v \in V$, $\beta^{-1}(v)$ consists of a single point.

(c) The restriction of β to $\tilde{V} \setminus E$ is a diffeomorphism onto its image $V \setminus \{0\}$.

(d) E is a submanifold of $\tilde{V}$ of codimension 1, which is diffeomorphic to $\mathbf{P}V$, and its tangent space at $([v], 0) \in E$ is $T_{([v],0)}E = \ker(T_{([v],0)}\beta)$.

(e) Let M be a $k+1$-manifold and $f\colon M \to V$ a smooth map with the property that $A = f^{-1}(0)$ is a submanifold of M of codimension 1 with tangent space $T_aA = \ker(T_af)$ for all $a \in A$. Then there exists a unique continuous map $\tilde{f}\colon M \to \tilde{V}$ such that $\beta \circ \tilde{f} = f$. This map $\tilde{f}$ is smooth, and its value at $a \in A$ is $\tilde{f}(a) = ([T_af(\xi)], 0)$, where $\xi \in T_aM$ is any tangent vector *not* in T_aA. (After choosing a suitable chart at a, we may assume that M is an open subset of $\mathbf{R}^{k+1}$ with coordinates $x_0, x_1, \ldots, x_k$, that a is the origin $a = 0$, that A is given by $x_0 = 0$, and that $\xi = (1, 0, 0, \ldots, 0)$. Now choose a basis $\mathscr{B} = (e_0, e_1, \ldots, e_n)$ of V with $e_0 = T_a f(\xi)$ and work in the corresponding coordinates $(u_1, u_2, \ldots, u_n)$ on $U = U_{\mathscr{B}}$, and $(v_0, v_1, \ldots, v_n)$ on V. Writing f in components, $f(x) = (f_0(x), f_1(x), \ldots, f_n(x))$, we have $v_i = f_i(x)$, so $u_i = f_i(x)/f_0(x)$, etc. Now apply the version of l'Hôpital's rule from the last homework.)

The manifold $\tilde{V}$ is also called the *(real) blow-up* of V at the origin and β the *blow-down map*. The blow-up of $\mathbf{R}^2$ at the origin is an infinitely wide Möbius strip. The exceptional fibre is the central circle of the Möbius strip. The blow-down map collapses the central circle to a point. More about blowing up later.