Math 6520 Homework due Friday 28 September 2018

1. We often call a right inverse of a map a *section* and a left inverse a *retraction*. Let $f\colon M \to N$ be a smooth map. Prove the following assertions.

(a) f is an immersion at $a \in M$ if and only if there exist open neighbourhoods U of a and V of $f(a)$ with the property that $f(U)$ is contained in V and $f\colon U \to V$ has a smooth retraction $r\colon V \to U$.

(b) Immersions are locally injective.

(c) f is a submersion at $a \in M$ if and only if there exist an open neighbourhood U of a with the property that $f(U)$ is open and a smooth section $s\colon f(U) \to U$ of f which maps $f(a)$ to a.

(d) Submersions are open maps.

2. Let I be the open interval $(-1,\infty)$ and let $f: I \to \mathbf{R}^2$ be the map $f(t) = (3at/(1+t^3, 3at^2/(1+t^3))$. Prove the following assertions.

(a) f is an injective immersion.

(b) f is not an embedding and $f(I)$ (with the subspace topology inherited from $\mathbf{R}^2$) is not a topological (let alone a smooth) manifold. (Draw a picture!)

(c) However, an injective immersion $f: M \to N$ of a compact manifold M into a Hausdorff manifold N is an embedding.

A continuous map π from a topological space Y to a topological space X is called a *local homeomorphism* if every point in Y has an open neighbourhood U with the property that $\pi(U)$ is open and $\pi\colon U \to \pi(U)$ is a homeomorphism. (A special case of a local homeomorphism is that of a *covering map* $\pi\colon Y \to X$, which is defined by the following property: every point in X has an open neighbourhood V such that the preimage $\pi^{-1}(V)$ is a union of disjoint open subsets of Y, each of which is mapped by π homeomorphically onto V.)

3. Let M be a manifold, $\tilde{M}$ a topological space, and $\pi\colon \tilde{M} \to M$ a local homeomorphism. Prove that $\tilde{M}$ has a unique smooth structure with the property that π is a submersion. Show that with respect to this smooth structure the map π is in fact étale (i.e. a local diffeomorphism). If π is surjective and if $f\colon \tilde{M} \to \tilde{M}$ and $g\colon M \to M$ are two continuous maps with the property that $g(\pi(y)) = \pi(f(y))$ for all y in $\tilde{M}$, show that f is smooth if and only g is smooth.

4. Let M be a submanifold of $\mathbf{R}^n$. Suppose that M contains the origin and let E be the tangent space to M at the origin. Choose a linear subspace F of $\mathbf{R}^n$ which is complementary to E and identify $E \times F$ with $\mathbf{R}^n$ via the linear isomorphism $(x,y) \mapsto x+y$. Prove the following statements.

(a) There exist open sets U in E and V in F and a smooth map $g\colon U \to V$ which satisfy the following conditions: $0 \in U$, $0 \in V$, and $M \cap (U \times V) = \operatorname{graph}(g)$.

(b) The germ of g at the origin is uniquely determined in the following sense: if $g_1\colon U_1 \to V_1$ and $g_2\colon U_2 \to V_2$ are two maps satisfying the condition in (a), then $g_1 = g_2$ on a neighbourhood of the origin contained in $U_1 \cap U_2$.

(c) The map g in (a) satisfies $g(0) = 0$ and $Dg(0) = 0$.