Math 6520 Homework due Friday 5 October 2018

1. Let M be an n-dimensional manifold. We say that M is *parallelizable* if there exist n smooth vector fields $\xi_1, \xi_2, \dots, \xi_n$ on M with the property that for all $a \in M$ the tangent vectors $\xi_{1,a}, \xi_{2,a}, \dots, \xi_{n,a}$ span the tangent space to M at a. Prove that M is parallelizable if and only if there exists a diffeomorphism $f\colon TM \to M \times \mathbf{R}^n$ with the following properties: $\mathrm{pr}_1 \circ f = \pi_M$, i.e. the following diagram commutes:

$$\begin{tikzcd} TM \arrow[rr, "f"] \arrow[dr, "\pi_M"'] & & M \times \mathbf{R}^n \arrow[dl, "\mathrm{pr}_1"] \\ & M, & \end{tikzcd}$$

and for each $a \in M$ the restriction of f to the fibre T_aM is a linear map from T_aM to $\{a\} \times \mathbf{R}^n$.

2. A *finite-dimensional real division algebra* is a pair $(A, \cdot)$, where A is a finite-dimensional real vector space and $\cdot\colon A \times A \to A$ is an $\mathbf{R}$-bilinear map called "multiplication", which is required to have no zero divisors ($x \cdot y = 0$ implies $x = 0$ or $y = 0$). (The multiplication is *not* required to be associative or commutative or to have a unit.) Prove that if $\mathbf{R}^n$ has the structure of a division algebra, then the sphere $\mathbf{S}^{n-1}$ is parallelizable.

3. Prove the following assertions.
   (a) The tangent bundle $T\mathbf{S}^{n-1}$ of the sphere is diffeomorphic to
   $$\{ (x, y) \in \mathbf{R}^{2n} \mid x \in \mathbf{S}^{n-1}, y \perp x \},$$
   where $\perp$ means "orthogonal with respect to the standard inner product".
   (b) $T\mathbf{S}^{n-1} \times \mathbf{R}$ is diffeomorphic to $\mathbf{S}^{n-1} \times \mathbf{R}^n$ for all $n \geq 0$.

4. Let ξ be the vector field on the real line $\mathbf{R}$ given by $\xi(x) = 1 + x^2$. Solve the initial value problem for ξ, i.e. solve the differential equation $x'(t) = 1 + x(t)^2$ with initial value $x(0) = x_0$. Here the unknown function $x\colon J \to \mathbf{R}$ is defined on an open interval J containing 0, which depends on the initial value x_0. Determine the maximal existence interval J for each x_0. Find the flow domain $\mathscr{D}$ and the flow $\theta\colon \mathscr{D} \to \mathbf{R}$ of ξ. Include a picture of some solutions $x(t)$ and of the domain $\mathscr{D}$.