Math 6520 Homework due Wednesday 17 October 2018

1 (derivations versus vector fields)**.** Let M be a manifold. A derivation of the $\mathbf{R}$-algebra $C^\infty(M)$ is called a *(global) derivation of* M. The module of derivations of M is denoted by $\mathrm{Der}(M)$. In other words, $\mathrm{Der}(M) = \mathrm{Der}_{\mathbf{R}}(C^\infty(M))$, the set of $\mathbf{R}$-linear maps $l\colon C^\infty(M) \to C^\infty(M)$ satisfying $l(fg) = l(f)g + fl(g)$. For a point a in M we have the algebra $C^\infty_{M,a}$ of germs at a and the module of *(pointwise) derivations at* a defined by $\mathrm{Der}_a(M) = \mathrm{Der}_{\mathbf{R}}(C^\infty_{M,a}, \mathbf{R})$, i.e. the set of $\mathbf{R}$-linear maps $l\colon C^\infty_{M,a} \to \mathbf{R}$ satisfying $l([fg]_a) = l([f]_a)g(a) + f(a)l([g]_a)$. Let $\mathscr{T}(M)$ be the set of smooth vector fields on M, i.e. smooth sections of the tangent bundle projection $\pi\colon TM \to M$. In class we defined maps

$$\mathscr{L} = \mathscr{L}_M\colon \mathscr{T}(M) \longrightarrow \mathrm{Der}(M), \quad \mathscr{L}_a = \mathscr{L}_{M,a}\colon T_aM \longrightarrow \mathrm{Der}_a(M)$$

by $\mathscr{L}(\xi) = df(\xi)$ and $\mathscr{L}_a(v) = d_af(v)$, and proved that $\mathscr{L}_a$ is an isomorphism of vector spaces for all a. We are going to show that $\mathscr{L}$ is an isomorphism as well by proving the following assertions.

(a) Let l be a global derivation and f a smooth function on M. If $f = 0$ on an open subset U, then $l(f) = 0$ on U. (Use bump functions.)

(b) Let l be a global derivation and let U be an open subset of M. There exists a unique derivation $l|_U \in \mathrm{Der}(U)$ with the property that $(l|_U)(f|_U) = l(f)|_U$ for all $f \in C^\infty(M)$. (We call $l|_U$ the *restriction* of l to U.)

(c) Let l be a global derivation and let a be a point in M. There exists a unique derivation $l_a \in \mathrm{Der}_a(M)$ with the property that $l_a([f]_a) = l(f)(a)$ for all $f \in C^\infty(M)$.

(d) $\mathscr{L}$ is injective.

(e) Let l be a global derivation and define $\xi\colon M \to TM$ by $\xi(a) = \mathscr{L}_a^{-1}(l_a)$. Then ξ is a smooth vector field. Conclude that $\mathscr{L}$ is surjective.

Let M be an n-manifold and a a point in M. We are going to blow up M at a. This is an operation which does not affect the manifold outside a, but replaces a with the collection of all possible tangent directions at a. Let I be an open interval in $\mathbf{R}$ and $c\colon I \to M$ a smooth path. The *velocity vector* $c'(t)$ at time $t \in I$ is defined by $c'(t) = T_tc(1)$. Here T_tc is the tangent map $T_t\mathbf{R} \to T_{c(t)}M$. As usual we identify $T\mathbf{R}$ with $\mathbf{R}\times\mathbf{R}$ and $T_t\mathbf{R} = \{t\}\times\mathbf{R}$ with $\mathbf{R}$, so that we can think of the number $1 \in \mathbf{R}$ as a tangent vector to $\mathbf{R}$ at any point. Thus $c'(t)$ is a vector in $T_{c(t)}M$. Now let $\bar{M}$ be the collection of all smooth paths $c\colon I \to M$ which are defined on an open interval I containing 0 and which have the property that $c'(0) \neq 0$. (The interval I may depend on c.) For two paths c_1 and c_2 in $\bar{M}$ define

$$c_1 \sim c_2 \iff \begin{cases} c_1(0) = c_2(0) & \text{if } c_1(0) \neq a \\ c_1(0) = c_2(0) \text{ and } c_1'(0) \text{ is proportional to } c_2'(0) & \text{if } c_1(0) = a. \end{cases}$$

The *blow-up* of M at a is the quotient $\mathrm{Bl}_a(M) = \bar{M}/{\sim}$ under this equivalence relation. The *blow-down map* $\beta\colon \mathrm{Bl}_a(M) \to M$ is defined by $\beta([c]) = c(0)$, where $[c]$ denotes the equivalence class of a path c. The *blow-up centre* is a, and the *central* or *exceptional fibre* is $E = \beta^{-1}(a)$.

2. Prove the following assertions. Of course, feel free to use results from previous homework, especially those on blowing up a vector space.

(a) Let V be a finite-dimensional real vector space. There is a bijection $\theta = \theta_V$ from $\mathrm{Bl}_0(V)$ to $\tilde{V}$ which satisfies $\tilde{\beta} \circ \theta = \beta$. (Here $\tilde{V}$ is the incidence relation in $\mathbf{P}V \times V$ and $\tilde{\beta} \colon \tilde{V} \to V$ the blow-down map defined in a previous homework problem.)

(b) Let U be an open subset of M. If U does not contain a, then the restriction of β to $\beta^{-1}(U)$ is a bijection $\beta \colon \beta^{-1}(U) \to U$. If U contains a, then the natural inclusion $\bar{U} \to \bar{M}$ induces a bijection $\mathrm{Bl}_a U \to \beta^{-1}(U)$. (We use this to identify $\mathrm{Bl}_a U$ with $\beta^{-1}(U)$.) There is a natural identification between the exceptional fibre and the projectivization of the tangent space $T_a M$.

(c) Let (U, ϕ) be a chart on M centred at a. The map $\phi \colon U \to \mathbf{R}^n$ lifts to a map $\tilde{\phi} \colon \mathrm{Bl}_a U \to \tilde{\mathbf{R}}^n$ satisfying $\tilde{\beta} \circ \tilde{\phi} = \phi \circ \beta$. If (V, ψ) is another chart centred at a, then $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a diffeomorphism from $\tilde{\beta}^{-1}(\phi(U \cap V))$ to $\tilde{\beta}^{-1}(\psi(U \cap V))$.

(d) There is a unique smooth structure on $\mathrm{Bl}_a(M)$ with the following property: for all charts (U, ϕ) on M which do not contain a, the map $\phi \circ \beta \colon \beta^{-1}(U) \to \phi(U)$ is a diffeomorphism; for all charts (U, ϕ) on M centred at a, the map $\tilde{\phi} \colon \mathrm{Bl}_a U \to \tilde{\beta}^{-1}(\phi(U))$ is a diffeomorphism.

(e) With respect to this smooth structure, β is smooth and E is a submanifold of $\mathrm{Bl}_a(M)$ of codimension 1 with tangent space $T_y E = \ker(T_y \beta)$ for all $y \in E$.

(f) The blow-up has the following universal property: let P be a manifold and $f \colon P \to M$ a smooth map which *cleanly intersects a* in the sense that $Q = f^{-1}(a)$ is a submanifold of P of codimension 1 with tangent space $T_q Q = \ker(T_q f)$ for all $q \in Q$. Then there exists a unique smooth map $\tilde{f} \colon P \to \mathrm{Bl}_a(M)$ such that $\beta \circ \tilde{f} = f$.

(g) Let A be a closed submanifold of M which contains a. The *proper* or *strict transform* of A, which is defined as the closure of $\beta^{-1}(A \setminus \{a\})$ in $\mathrm{Bl}_a(M)$, is a submanifold of $\mathrm{Bl}_a(M)$ and is diffeomorphic to $\mathrm{Bl}_a(A)$.