Math 6520 Homework due Wednesday 24 October 2018

1. Let A be a finite-dimensional associative algebra over $\mathbf{R}$ with unit element 1 and let $A^\times$ be its group of invertible elements. Prove the following statements.

(a) There exists a norm on A with the property that $\|xy\| \le \|x\|\,\|y\|$ for all x, $y \in A$.

(b) With respect to this norm, if $\|x\| < 1$, then $1 - x$ is invertible.

(c) $A^\times$ is an open submanifold of A, where we give A its standard smooth structure as a finite-dimensional real vector space.

(d) The group law on $A^\times$ is compatible with the smooth structure. Therefore $A^\times$ is a Lie group.

2. Let A be as in Exercise 1. Let $x \mapsto x^*$ be an *involution* of A, i.e. an anti-automorphism of A (linear map with the property $(xy)^* = y^*x^*$) satisfying $(x^*)^* = x$ for all x. An element x of A is *unitary* if $x^*x = 1$, *selfadjoint* if $x^* = x$, *anti-selfadjoint* if $x^* = -x$. The *unitary group* of A with respect to the involution is the set $\mathbf{U}(A)$ consisting of all unitary elements. Prove the following statements.

(a) The formula $g \cdot x = gxg^*$ defines an action of the Lie group $A^\times$ on the manifold A.

(b) The unitary group $\mathbf{U}(A)$ is a Lie subgroup of $A^\times$. Its tangent space at the identity is the set of all anti-selfadjoint elements of A.

(c) If there exists a norm on A with the property that $\|x^*x\| = \|x\|^2$ for all x, then $\mathbf{U}(A)$ is compact.

The classical Lie groups come in three families, which are associated with the real numbers, the complex numbers, and the quaternions. *Quaternions* are 2×2-matrices of the form

$$x = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix},$$

where a and b are in $\mathbf{C}$. The collection of quaternions is denoted by $\mathbf{H}$ in honour of their inventor, Hamilton. Ireland has an annual celebration of the invention of the quaternions on Hamilton Day, 16 October. The *(quaternionic) conjugate* of x is

$$x^* = \bar{x}^T = \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix},$$

where $\bar{x}$ denotes the entry-by-entry *complex* conjugate of the matrix x. In this homework we denote the imaginary unit in $\mathbf{C}$ by $\sqrt{-1}$ instead of i. We also define

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad k = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$$

and $\gamma(x) = i\bar{x}i^{-1} = -i\bar{x}i$ for $x \in M(2, \mathbf{C})$. We identify the real numbers with the copy of $\mathbf{R}$ consisting of all real multiples of $1 \in \mathbf{H}$.

A. **Lemma.** (a) *γ is an anti-linear (where "anti" means $\gamma(cx) = \bar{c}\gamma(x)$ for $c \in \mathbf{C}$) algebra homomorphism, and x is a quaternion if and only if $\gamma(x) = x$. Hence $\mathbf{H}$ is a real subalgebra of $M(2, \mathbf{C})$.*

(b) *We have $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The collection $\{1, i, j, k\}$ is a basis of $\mathbf{H}$ considered as a real vector space. The centre of $\mathbf{H}$ is $\mathbf{R}$.*

(c) *Quaternionic conjugation is an anti-automorphism of the* $\mathbf{R}$*-algebra* $\mathbf{H}$ *(meaning that it is* $\mathbf{R}$*-linear and satisfies* $(xy)^* = y^*x^*$*). We have* $x^*x \in \mathbf{R}$*. If* $x \neq 0$*, then* $x^*x > 0$ *and* x *has an inverse* $x^{-1} = (x^*x)^{-1}x^*$*.* $\mathbf{H}$ *satisfies all the field axioms except for the commutativity of multiplication.*

(d) *The set of* unit quaternions $\{\, x \in \mathbf{H} \mid x^*x = 1 \,\}$ *is a closed submanifold of* $\mathbf{H}$ *diffeomorphic to* $\mathbf{S}^3$ *and is a Lie group.*

We call $\mathbf{H}$ a *skew field* or a *unital associative real division algebra*. The quantity $|x| = \sqrt{x^*x}$ is the *norm* of $x \in \mathbf{H}$. It is a theorem of Frobenius that every finite-dimensional unital associative real division algebra is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. (If we weaken "associative" to "alternative" there is one more item on the list, namely the 8-dimensional algebra of *octonions* $\mathbf{O}$ found by Graves and Cayley. The octonions are associated with the *exceptional* Lie groups.) Before coming up with the quaternions Hamilton wasted considerable effort trying to construct a skew field that is three-dimensional over $\mathbf{R}$. From a later letter to his son Archibald:

> "Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only add and subtract them.' "

There is actually a very simple linear algebra argument showing that an odd-dimensional division algebra $\mathbf{A}$ over $\mathbf{R}$ must be isomorphic to $\mathbf{R}$. Namely, for each $a \in \mathbf{A}$ the left translation map $L_a \colon \mathbf{A} \to \mathbf{A}$ defined by $L_a(b) = ab$ is $\mathbf{R}$-linear, so has a real eigenvector v with real eigenvalue λ. Then $av = \lambda v$, so $a = \lambda$!

We can speak of column n-vectors with quaternionic coefficients and we denote the set of all such by $\mathbf{H}^n$. Similarly, we have the set of $m \times n$-matrices $M(m \times n, \mathbf{H})$ with quaternionic coefficients. We can multiply a quaternionic $m \times n$-matrix A by a quaternionic $n \times p$-matrix B to get a quaternionic $m \times p$-matrix $C = AB$. By expanding each quaternionic entry to a little 2×2-matrix we can view a quaternionic $m\times n$-matrix as a complex $2m\times2n$-matrix. In this way $M(m\times n, \mathbf{H})$ is a real subspace of $M(2m \times 2n, \mathbf{C})$, and the algebra of square quaternionic matrices $M(n, \mathbf{H})$ is a real subalgebra of $M(2n, \mathbf{C})$. For $X \in M(2n, \mathbf{C})$ define

$$\gamma(X) = \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix} \bar{X} \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix}^{-1} = -\begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix} \bar{X} \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix}.$$

B. **Lemma.** (a) *$X \in M(2n, \mathbf{C})$ is quaternionic if and only if $\gamma(X) = X$.*

(b) *If $X \in M(n, \mathbf{H})$, then the determinant of X (viewed as a complex $2n \times 2n$-matrix) is real.*

(c) *A matrix $X \in M(n, \mathbf{H})$ has an inverse in $M(n, \mathbf{H})$ if and only if* $\det X$ *is nonzero.*

3. Prove Lemma A.

4. Prove Lemma B.

5. Let $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. The *adjoint* of a matrix $X = (x_{ij}) \in M(n, \mathbf{F})$ is the conjugate transpose matrix $X^* = (\bar{x}_{ji})$, where $x \mapsto \bar{x}$ is the conjugation map in the algebra $\mathbf{F}$. The *unitary group in n dimensions* over $\mathbf{F}$ is the set $\mathbf{U}(n, \mathbf{F})$ consisting of all matrices X that satisfy $X^*X = I$. Show that $\mathbf{U}(n, \mathbf{F})$ is a compact Lie subgroup of $\mathbf{GL}(n, \mathbf{F})$ and compute its dimension.