Math 6520 Homework due Wednesday 31 October 2018

In this homework, "manifold" means "manifold in the weak sense", i.e. a set equipped with a smooth structure.

1. Let M be a closed submanifold of a paracompact manifold N and $f\colon M \to \mathbf{R}$ a smooth function. We proved in class that there exists a smooth function $\bar{f}\colon N \to \mathbf{R}$ such that $\bar{f}|_M = f$. Give a counterexample to this statement for a nonclosed submanifold M.

Let M be a manifold and R an equivalence relation on M. We say that the equivalence relation R is *regular* if there exists a smooth structure on the quotient set M/R with the property that the quotient map $p\colon M \to M/R$ is a submersion. We identify the relation R with its graph, i.e. the set of pairs $(a, b) \in M \times M$ with the property that a and b are congruent modulo R. We denote by $\pi_1\colon R \to M$ and $\pi_2\colon R \to M$ the projections of R onto the respective factors.

2. Suppose that R is a regular equivalence relation on a manifold M. Prove the following statements.
   (a) There is a *unique* smooth structure on M/R such that $p\colon M \to M/R$ is a submersion.
   (b) R is a submanifold of $M \times M$ and π_1 is a submersion. (Apply the regular value theorem.)
   (c) The equivalence classes of R are closed submanifolds of M.
   (d) M/R is Hausdorff if and only if R is a closed subset of $M \times M$ (whether M is Hausdorff or not).

It is a theorem of Godement that the condition of Problem 2 is sufficient for regularity; i.e. if R is a submanifold of $M \times M$ and π_1 is a submersion, then R is regular. For a proof of Godement's theorem see Part II, § III.12 of J.-P. Serre, *Lie algebras and Lie groups*, second ed., Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 2006, corrected fifth printing. In the next problem you may use this theorem without proof.

3. Let G be a Lie group, M a manifold, and $\theta\colon G \times M \to M$ a smooth left action. The *orbit relation* R on M is defined by declaring $a \in M$ to be equivalent to ga for all $g \in G$. The quotient set M/R is called the *orbit space* of the action and denoted by M/G. The action is *free* if for all $a \in M$ the stabilizer subgroup $G_a = \{\, g \in G \mid ga = a \,\}$ is the trivial subgroup $\{1\}$. The action is *proper* if the map $\Theta\colon G \times M \to M \times M$ defined by $\Theta(g, a) = (a, ga)$ is closed, i.e. maps closed sets to closed sets. Prove the following assertions.
   (a) If G is compact and M is Hausdorff and second countable, the action is proper.
   (b) If the action is free, then Θ is an immersion.
   (c) If the action is proper and free, then Θ is an embedding and for each $a \in M$ the orbit $Ga = \{\, ga \mid g \in G \,\}$ is a submanifold of M.
   (d) If the action is proper and free, then the orbit relation is regular. It follows that the orbit space M/G has a smooth structure with the property that the quotient map $M \to M/G$ is a submersion. Moreover, M/G is Hausdorff.

4. Let G be a Lie group and H a Lie subgroup. Prove that the set of cosets G/H has a smooth structure with the property that the quotient map $G \to G/H$ is a submersion.