Math 6520 Homework due Wednesday 7 November 2018

1. Let G be a Lie group and M a manifold. Let θ be a G-action on M, let a be a point in M, and let G_a denote its stabilizer. As usual we write $\theta(g,a) = ga = \theta_g(a) = \theta^a(g)$. Prove the following statements.

(a) The map $\bar{\theta}^a : G/G_a \to M$ which sends the coset $gG_a \in G/G_a$ to $\theta^a(g) \in M$ is well-defined and smooth (where G/G_a is given the smooth structure defined in Exercise 4 on the 2 Nov homework).

(b) $\bar{\theta}^a$ is an injective immersion and therefore the orbit Ga is an immersed submanifold of M.

(c) Prove that the following conditions are equivalent:

 (i) θ^a is a surjective submersion for some $a \in M$.

 (ii) θ^a is a surjective submersion for all $a \in M$.

 (iii) $\bar{\theta}^a : G/G_a \to M$ is a diffeomorphism.

Under any of the equivalent conditions in (c) we call M a *homogeneous space* or *homogeneous G-space*. We can reformulate the result of (c) as follows: if X is any set equipped with a transitive G-action such that the stabilizer G_x of any point $x \in X$ is closed, then there is a unique smooth structure on X which makes X a homogeneous G-space.

2. Let $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. We equip $\mathbf{F}^n$ with the *standard Hermitian inner product* $\langle u, v\rangle = u^*v = \sum_{j=1}^k u_j^* v_j$. This is a biadditive map $\langle\cdot,\cdot\rangle : \mathbf{F}^n \times \mathbf{F}^n \to \mathbf{F}$ with the following properties: for all $u, v \in \mathbf{F}^n$ and for all $\lambda, \mu \in \mathbf{F}$,

(i) $\langle u\lambda, v\mu\rangle = \lambda^*\langle u, v\rangle\mu$,

(ii) $\langle v, u\rangle = \langle u, v\rangle^*$,

(iii) $\langle u, u\rangle \geq 0$ with equality only for $u = 0$.

(Here $\lambda \mapsto \lambda^*$ denotes conjugation in the skew field $\mathbf{F}$. We have $\lambda = \lambda^*$ if and only if λ is real, so the quantity $\langle u, u\rangle$ is real for all u.)

A *k-frame* in $\mathbf{F}^n$ is an ordered k-tuple of linearly independent vectors in $\mathbf{F}^n$. We denote the set of k-frames in $\mathbf{F}^n$ by $\mathrm{Fr}_k(\mathbf{F}^n)$. A k-frame x is called *orthonormal* if $\langle x_i, x_j\rangle = \delta_{i,j}$ for $i, j = 1, 2, \dots, k$. The set of orthonormal k-frames in $\mathbf{F}^n$ is called a *Stiefel manifold* and denoted by $V_k(\mathbf{F}^n)$. The main purpose of this exercise is to show that the Stiefel manifold is a manifold.

(a) Identifying a k-frame $(x_1, x_2, \dots, x_k)$ with the matrix $x \in M_{n\times k}(\mathbf{F})$ whose columns are $x_1, x_2, \dots, x_k$, show that $\mathrm{Fr}_k(\mathbf{F}^n)$ is open in $M_{n\times k}(\mathbf{F})$.

(b) Show that a k-frame x is orthonormal if and only if $x^*x = 1_k$, the $k \times k$-identity matrix.

(c) Show that the Stiefel manifold $V_k(\mathbf{F}^n)$ is a compact submanifold of $\mathrm{Fr}_k(\mathbf{F}^n)$. (Define a suitable action of the unitary group $\mathbf{U}(n, \mathbf{F})$ on $\mathrm{Fr}_k E$ one of whose orbits is the Stiefel manifold.)

(d) Show that the Stiefel manifold $V_k(\mathbf{F}^n)$ is diffeomorphic to $\mathbf{U}(n, \mathbf{F})/K$, where K is a closed subgroup of $\mathbf{U}(n, \mathbf{F})$ isomorphic to $\mathbf{U}(n - k, \mathbf{F})$.

3. Let $\pi : E \to B$ be a smooth real vector bundle. Let $\sigma \in \Gamma(E)$ be a smooth section satisfying $\sigma(b_0) = 0$ for a certain b_0 in B. Show that σ can be written as $\sigma = \sum_{i=1}^p f_i\sigma_i$ with $\sigma_i \in \Gamma(E)$ and coefficients $f_i \in C^\infty(B)$ satisfying $f_i(b_0) = 0$. (Here p is not necessarily equal to the rank of E.)