Math 6520 Homework due Wednesday 14 November 2018

1. A smooth map $f\colon M \to N$ is *transverse* to a submanifold B of N, notation $f \pitchfork B$, if $T_a f(T_a M) + T_{f(a)}B = T_{f(a)}N$ for all $a \in A$, where $A = f^{-1}(B)$. Show that A is a submanifold of M of dimension $\dim(A) = \dim(M) - \dim(N) + \dim(B)$ if f is transverse to B.

2. Two smooth maps $f\colon X \to Z$ and $g\colon Y \to Z$ are *transverse* if $T_x f(T_x X) + T_y g(T_y Y) = T_z Z$ for all $x \in X$, $y \in Y$ and $z \in Z$ such that $f(x) = g(y) = z$. Notation: $f \pitchfork g$. The *fibred product* of X and Y over Z (with respect to f and g) is the subset of $X \times Y$ defined by

$$X \times_Z Y = \{\, (x, y) \in X \times Y \mid f(x) = g(y) \,\}.$$

Prove the following assertions.

(a) f and g are transverse if and only $f \times g\colon X \times Y \to Z \times Z$ is transverse to the diagonal Δ_Z.

(b) If f and g are transverse, then $X \times_Z Y$ is a submanifold of $X \times Y$ of dimension $\dim(X) + \dim(Y) - \dim(Z)$. For $(x, y) \in X \times_Z Y$ with $f(x) = g(z) = z$ we have $T_{(x,y)}(X \times_Z Y) = T_x X \times_{T_z Z} T_y Y$.

Let $I \subseteq \mathbf{R}$ be an open interval. A *time-dependent* smooth vector field on a manifold M is a smooth map $\xi\colon I \times M \to TM$ such that $\xi(t, a)$ is in $T_a M$ for all a in M and t in I. Equivalently, ξ is a smooth vector field on the product $I \times M$ with the property that $\xi(t, a)$ is tangent to $\{t\} \times M$ for all t, i.e. ξ does not involve $\partial/\partial t$. The *ordinary differential equation* for a time-dependent smooth vector field ξ with *starting time* $t_0 \in I$ and *initial value* $a_0 \in M$ is the initial-value problem

$$(IVP_{t_0,a_0}) \qquad \frac{d}{dt}\gamma(t) = \xi(t, \gamma(t)), \qquad \gamma(t_0) = a_0$$

for smooth curves $\gamma\colon J \to M$ defined on some open interval J which contains t_0 and is contained in I. (This is a so-called *non-autonomous* ODE. By contrast, ordinary vector fields are often called *time-independent* and the associated ODE *autonomous*.) Define a time-*in*dependent vector field $\hat{\xi}$ on $I \times M$ by $\hat{\xi}(t, a) = (t, 1, \xi(t, a))$. (Here we identify $T_{t,a}(I \times M)$ with $I \times \mathbf{R} \times T_a M$.) Then we have the (autonomous) initial value problem for $\hat{\xi}$ with initial value $\hat{a}_0 = (t_0, a_0)$:

$$(\widehat{IVP_{t_0,a_0}}) \qquad \frac{d}{dt}\hat{\gamma}(t) = \hat{\xi}(\hat{\gamma}(t)), \qquad \hat{\gamma}(t_0) = (t_0, a_0)$$

(This is a standard trick for reducing non-autonomous ODE to autonomous ODE.) Below you may use the following properties.

(a) Solutions of $(\widehat{IVP_{t_0,a_0}})$ are of the form $\hat{\gamma}(t) = (t, \gamma(t))$, where $\gamma(t)$ is a solution of (IVP_{t_0,a_0}).

(b) Therefore, by the flow theorem for time-independent vector fields, the problem (IVP_{t_0,a_0}) has a smooth solution $\gamma\colon J \to M$ defined on an open interval J which contains t_0. The interval J is contained in I, because $\xi(t, a)$ is not defined for t not in I. By the uniqueness part of the flow theorem, if $\gamma'\colon J' \to M$ is another solution, then $\gamma = \gamma'$ on $J \cap J'$. It follows that (IVP_{t_0,a_0}) has a solution defined on a maximal open interval, which may depend on t_0 and a_0.

3. Let G be a Lie group, I an open interval, and $\delta\colon I \longrightarrow T_1G$ a smooth path in the tangent space T_1G, where 1 is the unit element of G.

(a) For $t \in I$ and $g \in G$ define $\delta_R(t, g) = T_1R_g(\delta(t))$. Show that δ_R is a smooth time-dependent vector field on G.

(b) Let J be a subinterval of I and $t_0 \in J$. Let g_0 and g be elements of G. Let $\zeta\colon J \longrightarrow G$ be a solution of the initial value problem for δ_R with initial value $\zeta(t_0) = g_0$. Show that $R_g \circ \zeta\colon J \longrightarrow G$ is a solution of the initial value problem with initial value $\zeta(t_0) = g_0g$.

(c) Show that for every starting time $t_0 \in I$ and every initial value $g_0 \in G$ the initial value problem for δ_R has a solution $\zeta\colon I \longrightarrow G$ defined on the entire interval I.

4. Let G be a Lie group and $\theta\colon G \times M \longrightarrow M$ a smooth left action. Let $\delta\colon I \longrightarrow T_1G$ be as in Exercise 3.

(a) For $a \in M$ and $t \in I$ define $\xi(t, a) = T_1\theta^a(\delta(t))$. Show that ξ is a smooth time-dependent vector field on M.

(b) Show that for every initial value $a_0 \in M$ and starting time $t_0 \in I$ the initial value problem for ξ has a solution $\gamma\colon I \longrightarrow M$ defined on the entire interval I. (Use the paths $\zeta\colon I \longrightarrow G$ found in Exercise 3.)

5. Let I be an open interval and $A\colon I \longrightarrow M(n, \mathbf{R})$ a smooth path in the vector space of real $n \times n$-matrices. Show that for every $t_0 \in I$ and $x_0 \in \mathbf{R}^n$ the initial value problem

$$\frac{dx}{dt}(t) = A(t)x(t), \qquad x(t_0) = x_0$$

has a solution $x\colon I \longrightarrow \mathbf{R}^n$ defined on the entire interval I. Solve this initial value problem explicitly for $n = 1$. Discuss why your solution may fail for $n \geq 2$.