Math 6520 Homework due Monday 3 December 2018

1. Let B be a manifold and $\pi\colon E \to B$ a real vector bundle.

(a) Assume that E is a tensor bundle, i.e. $E = \mathscr{F}(TB)$, where $\mathscr{F}$ is a smooth (say, contravariant) functor from the category of finite-dimensional real vector spaces to itself. Then sections s of E can be pulled back along smooth maps $\phi\colon B \to B$, namely $\phi^*(s)(b) = \mathscr{F}(T_b\phi) \circ s \circ \phi(b)$ for $b \in B$. The *Lie derivative* $\mathscr{L}_\xi(s)$ of a smooth section s along a smooth vector field ξ on B is defined by $\mathscr{L}_\xi(s) = \frac{d}{dt}\theta_t^*(s)|_{t=0}$, where θ is the flow of ξ. Show that $\mathscr{L}_\xi(fs) = \mathscr{L}_\xi(f)s + f\mathscr{L}_\xi(s)$ for every smooth function $f\colon B \to \mathbf{R}$.

(b) Show that for every $b \in B$ and $h \in E_b$ there exists a smooth section $s \in \Gamma(B, E)$ with the property that $s(b) = h$.

(c) Let $\mathbf{k} = C^\infty(B)$ be the ring of smooth functions on B. Let E^* be the dual bundle to E and let $\Gamma(B, E)^* = \operatorname{Hom}_{\mathbf{k}}(\Gamma(B, E), \mathbf{k})$ be the $\mathbf{k}$-module dual to $\Gamma(B, E)$. Define a $\mathbf{k}$-bilinear pairing

$$P\colon \Gamma(B, E^*) \times \Gamma(B, E) \longrightarrow \mathbf{k}$$

by $P(\sigma, s)(b) = \sigma(b)(s(b))$ for $b \in B$. Let $P^\sharp\colon \Gamma(B, E^*) \to \Gamma(B, E)^*$ be the $\mathbf{k}$-linear map defined by $P^\sharp(\sigma)(s) = P(\sigma, s)$. Prove that $P^\sharp$ is an isomorphism.

2. Let V be a finite-dimensional real vector space and let $\alpha \in A^k(V)$. We call α *decomposable* if $\alpha \neq 0$ and $\alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ for some $v_1, v_2, \ldots, v_k \in V$. The *annihilator* of α is the linear subspace $\alpha^\perp = \{\, v \in V \mid \alpha \wedge v = 0 \,\}$ of V. Prove the following statements.

(a) Let $\alpha \neq 0$. Then $\dim(\alpha^\perp) = k$ if and only if α is decomposable.

(b) Every k-dimensional subspace W of V is of the form $W = \alpha^\perp$ for a decomposable element $\alpha \in A^k(V)$, which is uniquely determined by W up to a multiplicative constant.

(c) Let $\beta \in A^l(V)$. If α and β are decomposable, then $\alpha^\perp \subseteq \beta^\perp$ if and only if $\beta = \alpha \wedge \gamma$ for some $\gamma \in A(V)$.

(d) If α and β are decomposable, then $\alpha^\perp \cap \beta^\perp = \{0\}$ if and only if $\alpha \wedge \beta \neq 0$, and in that case $(\alpha \wedge \beta)^\perp = \alpha^\perp \oplus \beta^\perp$.

(e) Every nonzero element of $A^{n-1}(V)$ is decomposable, where $n = \dim(V)$.

Let $\xi\colon I \to TM$ be a time-dependent vector field on a manifold M, where I is an open interval containing 0. Let $\hat{\xi} = \frac{\partial}{\partial t} + \xi$ be the associated time-independent vector field on $I \times M$, and let $\hat{\theta}\colon \hat{\mathscr{D}} \to I \times M$ be the flow of $\hat{\xi}$, where $\hat{\mathscr{D}} \subseteq \mathbf{R} \times I \times M$ is the flow domain of $\hat{\xi}$. Write points in $\mathbf{R} \times I \times M$ as (s, t, x). Since the first component of $\hat{\xi}$ is $\frac{\partial}{\partial t}$, the flow is of the form $\hat{\theta} = (s + t, \theta(s, t, x))$ for a unique smooth map $\theta\colon \hat{\mathscr{D}} \to M$. The path $\gamma(s) = \theta(s - t, t, x)$ is the trajectory of ξ passing through x at time $s = t$.

In particular, $\gamma(s) = \theta(s, 0, x)$ is the trajectory of ξ passing through x at time $s = 0$. In the sequel we abbreviate $\theta(s, 0, x)$ to $\theta(s, x)$ and refer to θ as the *flow* of ξ. The domain $\mathscr{D} \subseteq \mathbf{R} \times M$ of θ consists of all (s, x) with $(s, 0, x) \in \hat{\mathscr{D}}$ and is therefore an open subset of $\mathbf{R} \times M$.

3 (Connected manifolds are homogeneous under the diffeomorphism group). Let M be a manifold. Prove the following assertions.

(a) M is connected if and only if it is C^∞ path-connected (the definition of which is left to your imagination).

(b) Assume M to be connected. For all x and y in M there exists a diffeomorphism $f\colon M \to M$ such that $f(x) = y$. (Choose a smooth path $c\colon [0,1] \to M$ connecting x to y. Show there exists a compactly supported time-dependent vector field ξ on M such that $\xi(t, c(t)) = c'(t)$ for $0 \leq t \leq 1$, and use the flow $(t,x) \to \theta(t,x)$ of ξ.)

A *time-dependent* differential form on a manifold M is a smooth function $\alpha\colon I \times M \to A(T^*M)$ such that $\alpha(t,x) \in A(T^*_x M)$ for all $(t,x) \in I \times M$, i.e. a form on $I \times M$ that does not involve dt. Here I is an open interval. For a time-dependent form α we define forms $\alpha_t \in \Omega(M)$ for each $t \in I$ by $\alpha_t(x) = \alpha(t,x)$.

4. Let α be a time-dependent form and $\xi\colon I \times M \to TM$ a time-dependent vector field. Let $(t,x) \mapsto \theta(t,x) = \theta_t(x)$ be the flow of ξ. Then

$$\frac{d}{dt}\theta_t^*\alpha_t = \theta_t^*(\mathscr{L}(\xi_t)\alpha_t + \dot{\alpha}_t),$$

where the dot denotes differentation with respect to t.

For a module V over a commutative ring $\mathbf{k}$ we let $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ be the dual module and for each $u \in V$ we define a map $\iota(u)\colon A^k(V^*) \to A^{k-1}(V^*)$ by

$$\iota(u)(\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k) = \sum_{l=1}^{k}(-1)^{l+1}\phi_l(u)\,\phi_1 \wedge \phi_2 \wedge \cdots \wedge \widehat{\phi_l} \wedge \cdots \wedge \phi_k$$

for $\phi_1, \phi_2, \ldots, \phi_k \in V^*$. Under the canonical isomorphism $A^k(V^*) \cong \mathrm{Alt}^k_{\mathbf{k}}(V)$ we have

$$(\iota(u)\phi)(v_1, v_2, \ldots, v_{k-1}) = \phi(u, v_1, v_2, \ldots, v_{k-1})$$

for $\phi \in \mathrm{Alt}^k_{\mathbf{k}}(V)$. Applying this to the ring of smooth functions $\mathbf{k} = C^\infty(M)$ of a manifold M and the module of smooth vector fields $V = \mathscr{T}(M)$ we get an operator $\iota(\xi)\colon \Omega^k(M) \to \Omega^{k-1}(M)$ for each vector field ξ. There are various relationships among the operators $\iota(\xi)$, $\mathscr{L}(\xi)$, and d, the most important of which is *É. Cartan's magic formula*, which says that $\mathscr{L}(\xi)$ is the graded commutator of d and $\iota(\xi)$,

$$\mathscr{L}(\xi) = [d, \iota(\xi)] = d\iota(\xi) + \iota(\xi)d.$$

5. Let μ be a *volume form* on an n-manifold with boundary M, that is a nowhere vanishing n-form ($\mu(x) \neq 0$ for all $x \in M$). Let ξ be a (time-independent) vector field. Then $\mathscr{L}(\xi)\mu$ is an n-form, so there is a unique function $\mathrm{div}(\xi)$, called the *divergence* of ξ relative to μ, satisfying $\mathscr{L}(\xi)\mu = \mathrm{div}(\xi)\mu$. Prove the following statements.

(a) $\mathrm{div}(\xi) = 0$ if and only if $\theta_t^*\mu = \mu$ for all t, where θ is the flow of ξ.

(b) For compact M we have the *divergence theorem*,

$$\int_M \mathrm{div}(\xi)\mu = \int_{\partial M} \iota(\xi)\mu.$$

(Here we use the orientation of M compatible with μ, i.e. an ordered basis $(v_1, v_2, \ldots, v_n)$ of $T_x M$ is positively oriented if $\mu(v_1 \wedge v_2 \wedge \cdots \wedge v_n) > 0$.) (Use Cartan's formula.)

(c) Now let ξ be a time-dependent vector field on M. Assume ξ is complete with flow θ. Let f be a time-dependent smooth function on M (i.e. a smooth function $f\colon I \times M \to \mathbf{R}$) and A a relatively compact open subset. Then we have the *transport equation*,

$$\frac{d}{dt}\int_{\theta_t(A)} f_t\,\mu = \int_{\theta_t(A)} \big(\dot{f}_t + \mathscr{L}(\xi_t)f_t + f_t\,\mathrm{div}(\xi_t)\big)\,\mu.$$