MATH 321 Manifolds and Differential Forms (II)

Homework 1 Solution

Due September 6, 3:00 p.m.

1.1 (4 points) Solution: This surface X is described by equation $F(x,y,z) = 0$, where $F(x,y,z) = x^2y - z^2$. So for any point $\mathbf{p} = (x_0, y_0, z_0) \in X$, the normal vector $\mathbf{v} = 0$ at $\mathbf{p}$ has the same direction as $(F'_x(p), F'_y(p), F'_z(p)) = (2x_0y_0, x_0^2, -2z_0)$. So $\mathbf{v} = 0$ if and only if $x_0 = 0$, $z_0 = 0$ while y_0 can be arbitrary. This is the y-axis. So y-axis is all of the singularities.

(See the illustration below. Note the y-axis is not shown, so we get a hole over there.) $\square$

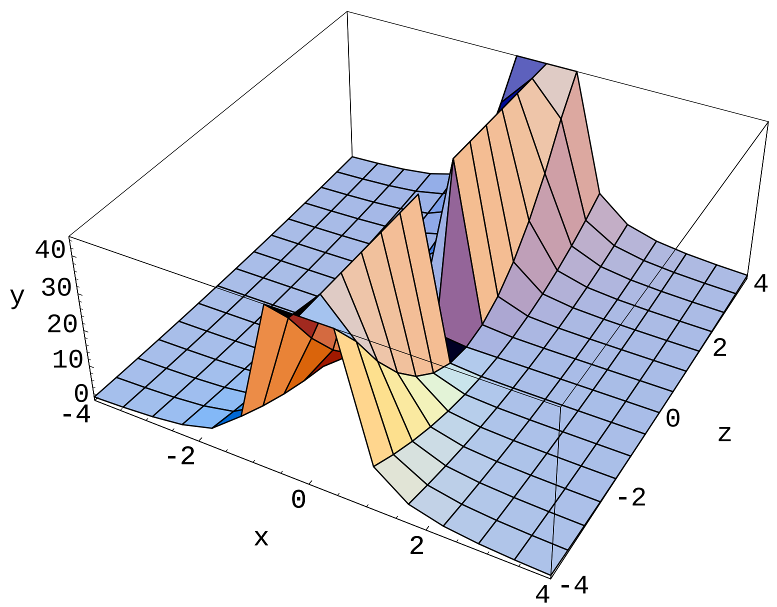

Figure 1: Graph of the surface $y = z^2/x^2$

1.2 (4 points) Solution: The surface X is defined by $F(\mathbf{x}) = 0$ where $F(\mathbf{x}) = ||\mathbf{x}||^2 - n = \sum_{i=1}^n x_i^2 - n$. The normal vector $\mathbf{v}$ at $\mathbf{a}$ has the same direction as $\text{grad}F(\mathbf{a}) = (2, \ldots, 2)$. So $\mathbf{x} \in T_aX$ if and only if $(\mathbf{x}-\mathbf{a}) \cdot \text{grad}F(\mathbf{a}) = 0$. Let Y be the solution space of $\mathbf{y} \cdot \text{grad}F(\mathbf{a}) = 0$, we then solve the linear equation $\sum_{i=1}^n y_i = 0$ and get $Y = \text{span}\{\mathbf{e_1} - \mathbf{e_2}, \mathbf{e_1} - \mathbf{e_3}, \ldots, \mathbf{e_1} - \mathbf{e_n}\}$ where $\mathbf{e_i}$'s are the standard basis vectors of $\mathbb{R}^n$. Let $\mathbf{v_i} = \mathbf{e_1} - \mathbf{e_{i+1}}$. Then $\mathbf{x} = \mathbf{a} + \sum_{i=1}^{n-1} t_i \mathbf{v_i}$ for some real coefficients $t_1, t_2, \ldots, t_{n-1}$. □

1.3 (4 points) Solution: For any $t \in \mathbb{R}$, $x(t)$ and $y(t)$ are well-defined and differentiable at t. Moreover, $x'(t) = 1 - \cos t$, $y'(t) = \sin t$. So all the singularities of X must satisfy $1 = \cos t$ and $\sin t = 0$, i.e. $t = 2k\pi$, $(k \in \mathbb{Z})$. We also discuss the following interesting points:

(1). The points at which tangent lines have slope 0. We then have $y'(t) = 0$, $x'(t) \neq 0$, which implies $t = \pi + 2k\pi (k \in \mathbb{Z})$.

(2). The points at which tangent line have slope 1: we then have $y'(t) = x'(t) \neq 0$. So we get $t = 2k\pi + \pi/2$, $(k \in \mathbb{Z})$.

(3). To let the slope be -1, we should have $t = 2k\pi - \pi/2$, $(k \in \mathbb{Z})$.

(4). To let the slope be ∞, we must have $x'(t) = 0$, $y'(t) \neq 0$, i.e. $\cos t = 1$, $\sin t \neq 0$. This is impossible. □

Remark: Note this parameterized curve is periodic with 2π as the period. The following illustration takes only one period $[0, 2\pi]$.

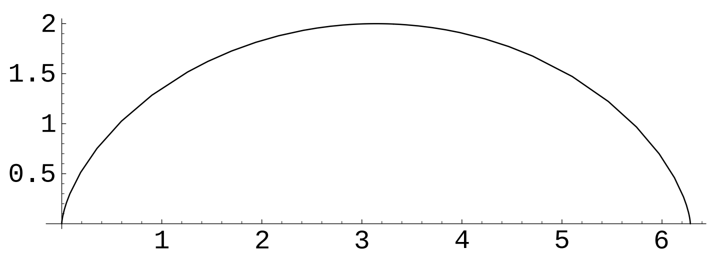

Figure 2: Graph of the parameterized curve in problem 1.3

1.5 (4 points) Solution: We define "a bicycle" as an object in 3-dimensional space consisting of the following pieces, a frame, which is abstracted as a triangle, a handle attached to one vertex of the frame which is able to rotate 360 degree in the plane perpendicular to the frame plane, and two wheels, with one attached to the handle, and the other attached to another vertex of the frame triangle.

To decide the degree of freedom of the bicycle, we choose three points A, B and C on the frame, which don't fall on a line. Then, for A, we have 3 degrees of freedom. Fix A, B can move on the sphere centered at A with AB as the radius. So B has 2 degree of freedom. Finally, with A and B fixed, C has only 1 degree of freedom, whose trajectory is a circle.

We choose a fourth point D on the handle, then it can move on a circle, so D has one degree of freedom. Finally, we choose one point E from the front wheel and one point F from the back wheel. Each of them should have one degree of freedom, as they can only make circular movement in a plane.

Add up all the above numbers, we decide a bicycle has 9 degrees of freedom. □

2.1 (4 points) Proof: $\forall y \in B_\varepsilon(x)$, and let $r = \varepsilon - ||y - x||$, we claim $B_r(y) \subset B_\varepsilon(x)$. Indeed, $\forall z \in B_r(y)$, we have $||z - x|| \leq ||z - y|| + ||y - x|| < \varepsilon$. Since z is arbitrarily chosen, we conclude $B_r(y) \subset B_\varepsilon(x)$. So by definition, $B_\varepsilon(x)$ is open. □