# MATH 321 Manifolds and Differential Forms (II)

# Homework 10 Solution

Due December 9, 3:00 p.m.

10.1 (3 points)

(i) Proof: $\int_{\partial M} \alpha = \int_M d\alpha = \int_M dxdy - dydx = 2\int_M dxdy = 2$(surface area of M). $\square$

(ii) Solution:

$$\begin{aligned}
Area &= \frac{1}{2}\int_{\partial M} \alpha = \frac{1}{2}\int_0^{2\pi} c^*\alpha \\
&= \frac{1}{2}\int_0^{2\pi} \cos^3 t 3\sin^2 t\cos t dt + \sin^3 t 3 cos^2 t \sin t dt \\
&= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t dt = \frac{3}{8}\int_0^{2\pi} (\sin 2t)^2 dt \\
&= \frac{3}{8}\int_0^{2\pi} \frac{1-\cos 4t}{2} dt \\
&= 3\pi/8
\end{aligned}$$

$\square$

(iii) Solution: $\int_{\partial M} \alpha = \int_M d\alpha = \int_M \sum_{i=1}^n dx = n$(Volume of M). $\square$

10.3 (6 points) Proof:

(i)

$$\begin{aligned}
df(*dg) &= (\sum_i \frac{\partial f}{\partial x_i} dx_i)(*\sum_j \frac{\partial g}{\partial x_j} dx_j) \\
&= \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} dx_i(*dx_j) \\
&= \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} dx \\
&= (grad(f)\cdot grad(g))\mu
\end{aligned}$$

$\square$

(ii)

$$\begin{aligned} d(*dg) &= d(\sum_i \frac{\partial g}{\partial x_i}(*dx_i)) \\ &= \sum_i [d(\frac{\partial g}{\partial x_i})(*dx_i) + \frac{\partial g}{\partial x_i} d(*dx_i)] \\ &= \sum_i [\sum_j \frac{\partial^2 g}{\partial x_i \partial x_j} dx_j(*dx_i)] \\ &= \sum_i \frac{\partial^2 g}{\partial x_i^2} dx = (\Delta g)\mu \end{aligned}$$

□

(iii) $d(f(*dg)) = df(*dg) + fd(*dg) = (gradf \cdot gradg)\mu + f(\Delta g)\mu = (gradf \cdot gradg + f\Delta g)\mu$. □

(iv) $\int_{\partial M} f(*dg) = \int_M d(f(*dg)) = \int_M (gradf \cdot gradg)\mu + f(\Delta g)\mu = D(f,g) + \int_M (f\Delta g)\mu$. □

(v) $\int_{\partial M}(f(*dg) - g(*df)) = D(f,g) + \int_M (f\Delta g)\mu - D(f,g) - \int_M (g\Delta f)\mu = \int_M (f\Delta g - g\Delta f)\mu$. □

(vi) LHS=$\int_{\partial M}(f(gradg) - g(gradf)) \cdot \mathbf{n}\mu_{\partial M} = \int_{\partial M}(f(*dg) - g(*df))\mu$, since $gradg \cdot \mathbf{n}\mu_{\partial M} = *dg$ and $gradf \cdot \mathbf{n}\mu_{\partial M} = *df$. To see this, note by Corollary 9.11, $\mu_{\partial M} = \mathbf{n} \cdot (*dx)$. So if we let $\mathbf{n} = (v_1, \cdots, v_n)$, then we have

$$\begin{aligned} gradg \cdot \mathbf{n}\mu_{\partial M} &= (gradg \cdot \mathbf{n}\mu)(\mathbf{n} \cdot (*dx)) \\ &= (\sum_i \frac{\partial g}{\partial x_i} v_i)(\sum_j (-1)^{j-1} dx_1 \cdots d\hat{x}_j \cdots dx_n v_j) \\ &= \sum_{i,j} (-1)^{j-1} v_i v_j \frac{\partial g}{\partial x_i} dx_1 \cdots d\hat{x}_j \cdots dx_n \\ &= \sum_i (-1)^{i-1} \frac{\partial g}{\partial x_i} dx_1 \cdots d\hat{x}_i \cdots dx_n \\ &= *dg \end{aligned}$$

□

10.4 (6 points)

(i) Proof: By Corollary 9.11, $\mu_{\partial B} = \mathbf{n} \cdot (*d\mathbf{x})$. This is the restriction of v to $\partial B$, where v is as in Exercise 9.5(ii). So $A_n(R) = \int_{\partial B} \mu_{\partial B} = \int_{\partial B} v|_{\partial B}$. □

(ii) Proof: $V_n(R) = \int_0^R \int_{\partial B_r} \mu_{\partial B_r} dr = \int_0^R An(r)dr$, where $B_r$ stands for the ball with radius r. The first equality is intuitively true and can be rigorously proved if you know the definition of Riemann integral. We shall not present a formal justification here, but will take it for granted henceforth. Therefore, we have $dV_n(R)/dR = A_n(R)$. By change of variable formula, we can deduce $V_n(R) = R^n V_n$. So $A_n(R) = dV_n(R)/dR = nR^{n-1}V_n$. In particular, for $R = 1$, we have $A_n = nV_n$. So $A_n(R)/A_n = nR^{n-1}V_n/nV_n = R^{n-1}$. Hence $A_n(R) = A_n R^{n-1}$. □

(iii) Proof:

$$\int_B g\mu_B = \int_0^R dr \int_{\partial B} f(r)\mu_{\partial B} = \int_0^R f(r)A_n(r)dr = A_n \int_0^R f(r)r^{n-1}dr$$

□

(iv) Proof: Let $f(r) = e^{-r^2}$, we get by (iii)

$$\int_B e^{-r^2}\mu_B = A_n \int_0^R r^{n-1}e^{-r^2}dr$$

Let $R \to \infty$, we get $\int_{\mathbb{R}^n} e^{-r^2}dx = A_n \int_0^\infty e^{-r^2}r^{n-1}dr$. Note

$$LHS = \prod_{i=1}^n \int_{-\infty}^\infty e^{-x_i^2}dx_i = (\int_{-\infty}^\infty e^{-r^2}dr)^n$$

we are then done. □

(v) Solution: By problem 2.6 (iii), and above (iv)

$$\pi^{n/2} = A_n \frac{1}{2}\Gamma(\frac{1+n-1}{2}) = A_n \frac{1}{2}\Gamma(n/2)$$

So, $A_n = 2\pi^{n/2}/\Gamma(n/2)$. For n=2m, by 2.6(ii), $A_{2m} = 2\pi^m/\Gamma(m) = 2\pi^m/(m-1)!$. For n=2m+1, by 2.6(i),

$$A_{2m+1} = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m-1)}$$

□

(vi) Solution: By $A_n = nV_n$ and (v) above, we get the formula for $V_n$. □

(vii) Proof: Proved in (ii). □

(viii) Solution: We just use the formula in (v) and (vi) to fill out the table. □

(ix) Solution: All the limits are zero. To see this, instead of using Stirling's formula, we choose to use the formulas developed in (v) and (vi), which is easier. E.g., to show $\lim_{n\to\infty} A_n = 0$ when n=2m,

$$A_{2m} = 2\pi(\frac{\pi}{1} \times \frac{\pi}{2} \times \frac{\pi}{3} \times \cdots \times \frac{\pi}{m-1})$$

The first three terms are greater than 1. All the other terms are smaller than 1, and get smaller and smaller. So $A_{2m} \to 0$ as $m \to \infty$. Other case can be proved similarly. □

11.6 (2 points) Proof: We parametrize $c_{\mathbf{x}}$ by $c_{\mathbf{x}} = t\mathbf{x}$, where $t \in [0,1]$, then

$$\int_{c_{\mathbf{x}}} \alpha = \int_0^1 c^*(\sum_{i=1}^n g_i dx_i) = \sum_{i=1}^n \int_0^1 g_i(t\mathbf{x}) x_i dt = \sum_{i=1}^n x_i \int_0^1 g_i(t\mathbf{x}) dt = \beta$$

□

11.7 (3 points)

$$\begin{aligned} \phi^*\alpha &= f(t\mathbf{x}) d(tx_{i_1}) \cdots d(tx_{i_k}) \\ &= f(t\mathbf{x})(x_{i_1}dt + tdx_{i_1}) \cdots (x_{i_k}dt + tdx_{i_k}) \\ &= f(t\mathbf{x})(\sum_{m=1}^k t^{k-1} dx_{i_1} \cdots (x_{i_m}dt) \cdots dx_{i_k} + t^k dx_I) \end{aligned}$$

So

$$\begin{aligned} \kappa\phi^*\alpha &= \sum_{m=1}^k \int_0^1 f(t\mathbf{x}) t^{k-1} x_{i_m} dt (-1)^{m+1} dx_{i_1} \cdots d\hat{x}_{i_m} \cdots dx_{i_k} \\ &= \sum_{m=1}^k (-1)^{m+1} (\int_0^1 f(t\mathbf{x}) t^{k-1} dt) x_{i_m} dx_{i_1} \cdots d\hat{x}_{i_m} \cdots dx_{i_k} \end{aligned}$$

□