MATH 321 Manifolds and Differential Forms (II)

Homework 2 Solution

Due September 13, 3:00 p.m.

2.4 (3 points) Solution: From $f(t\mathbf{x}) = t^k f(\mathbf{x})$, we can obtain by taking derivatives with respect to t on both sides that $\sum_i x_i \frac{\partial f(t\mathbf{x})}{\partial x_i} = kt^{k-1} f(\mathbf{x})$. Let $t = 1$, we are then done. $\square$

2.5 (5 points)

(i) Proof:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} \overset{x=1/h}{=} \lim_{h\to\infty} \frac{h}{e^{h^2}} = \lim_{h\to\infty} \frac{1}{2he^{h^2}} = 0$$

according to l'Hospital law. So, f is differential at 0, and $f'(0) = 0$ $\square$

(ii) Proof: We work by induction. First of all $f'(0) = 0$ and $f'(x) = \frac{2e^{-1/x^2}}{x^3}$ for $x \neq 0$. Then it's clear that $f \in C^1(\mathbb{R})$. Assume $f \in C^k(\mathbb{R})$, with $f^{(k)}(0) = 0$ and $f^{(k)}(x) = \frac{P_k(x)}{Q_k(x)} e^{-1/x^2}$ for $x \neq 0$, where P_k and Q_k are polynomials of x. Then we prove $f^{(k)}$ is continuously differentialable, $f^{(k+1)}(0) = 0$ and $f^{(k+1)}(x) = \frac{P_{k+1}(x)}{Q_{k+1}(x)} e^{-1/x^2}$ for $x \neq 0$, where P_{k+1} and Q_{k+1} are polynomials of x. Indeed, similar to the case of $k = 1$, by replacing $1/x$ with h and the fact that e^{h^2} increases faster than any polynomials of h when $h \to \infty$, we can conclude that the result desired. $\square$

(iii) Solution: See the illustration below.

2.6 (5 points)

(i) Proof:

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = -e^{-t} t^x|_0^\infty - \int_0^\infty (-e^{-t}) dt^x = \int_0^\infty e^{-t} x t^{x-1} dt = x\Gamma(x)$$

The second = is due to Leibnitz's law. $\square$

(ii) Proof: By (i), $\Gamma(n) = (n-1)\Gamma(n-1)$ for $n > 1$. By induction, $\Gamma(n) = (n-1)!\Gamma(1)$. As $\Gamma(1) = 1$, we conclude $\Gamma(n) = (n-1)!$ for positive integers n. $\square$

(iii) Proof:

$$\int_0^\infty e^{-u^2} u^\alpha du \overset{t=u^2}{=} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}} \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^\infty e^{-t} t^{\frac{\alpha+1}{2}-1} dt = \frac{1}{2}\Gamma(\frac{\alpha+1}{2})$$

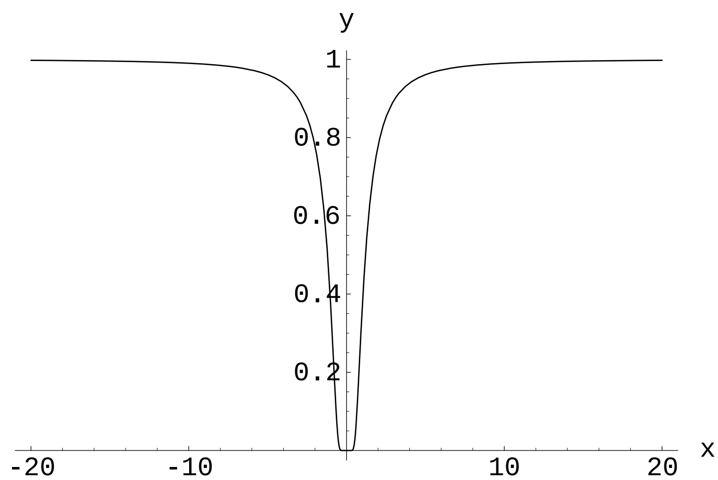

Figure 1: Graph of the function $f(x) = e^{-1/x^2}$

$\square$

2.7 (4 points)

(i) Proof: $\gamma = \Gamma(1/2) = 2\int_0^\infty e^{-u^2} du$ by 2.6 (iii) and by letting $\alpha = 0$. So, $\gamma = \int_{-\infty}^\infty e^{-u^2} du$. $\square$

(ii) Proof: By (i), $\gamma = \int_{-\infty}^\infty e^{-x^2} dx = \int_{-\infty}^\infty e^{-y^2} dy$. So $\gamma^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} dxdy$. $\square$

(iii) Proof: According to (ii) and polar coordinate, we have

$$\gamma^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} dxd = \int_0^{2\pi} \int_0^\infty e^{-r^2} rdrd\theta$$

$\square$

(iv) Proof: By (iii), $\gamma^2 = \pi$. So $\gamma = \sqrt{\pi}$. $\square$

1.7 Solution:

(i) We start with the case of a square. See figure 2.

A square can have three singular status, which is obtained by collapsing the whole square onto the straight line on which the edge AB lies. See figure 3.

We draw three nodes, marked by 1, 2 and 3, respectively, to stand for these three singular status. Please see figure 4 for the illustration.

For status I, we have two possible choices of moving the square, either as indicated by figure 5, or as indicated by figure 6.

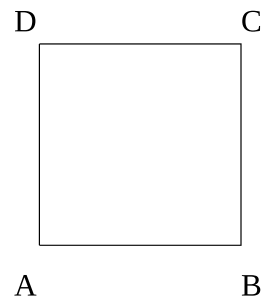

Figure 2:

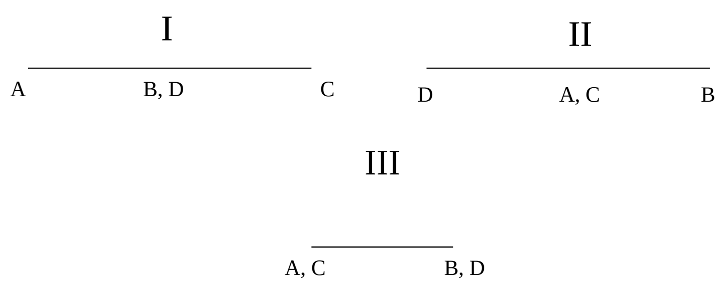

Figure 3:

So the node 1 should have two branches. One goes to 3 and the other goes to 2. See figure 7. (The arrows in the graph mean the direction of movement.)

Apply the same argument to node 2, we should get figure 8.

Finally, if we start from status III, we still have two choices. One is going to I and the other is going to II. So we get figure 9.

□

(ii) The case of a parallelogram which is not a square is similar to the case of square. The only difference is that the former case has only two singular status: I and II. The third singular status III will not appear as the lengths

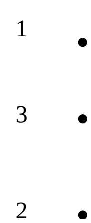

Figure 4:

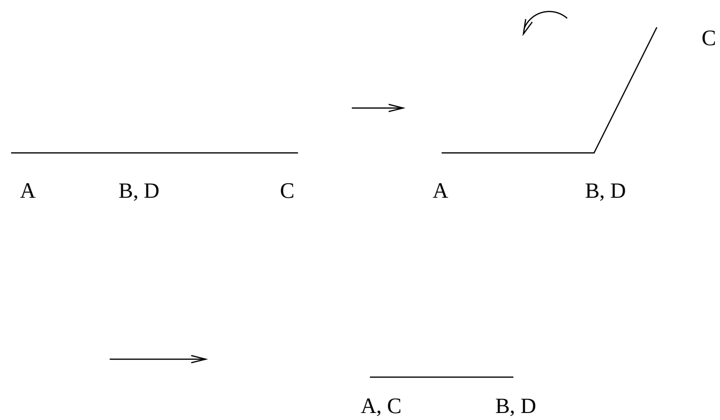

Figure 5:

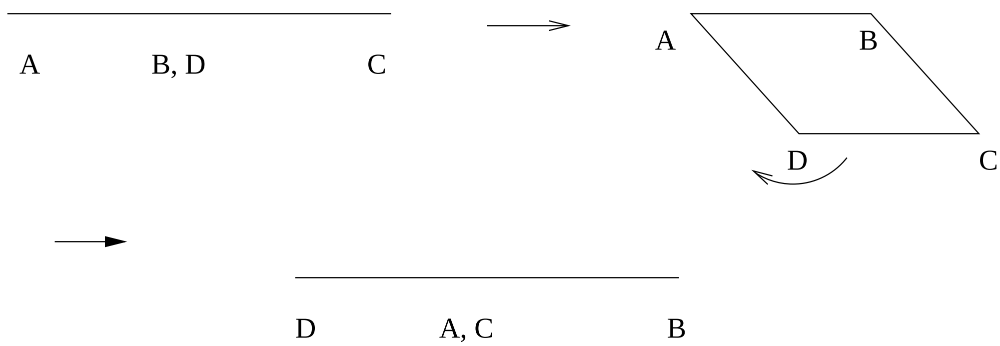

Figure 6:

of edges are not equal. So, by similar analysis, we should get the graph of the configuration space as indicated by figure 10.

□

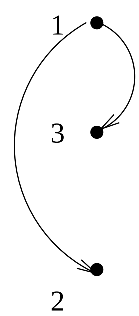

Figure 7:

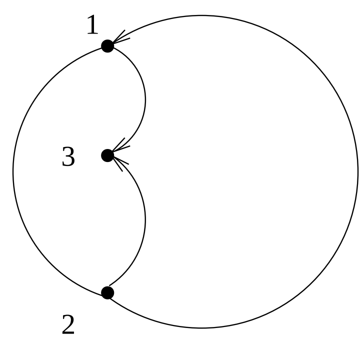

Figure 8:

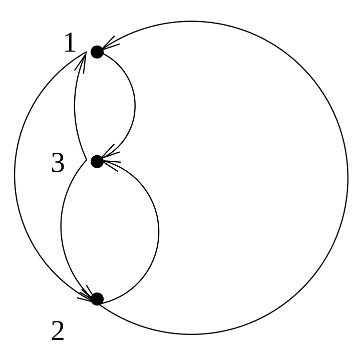

Figure 9:

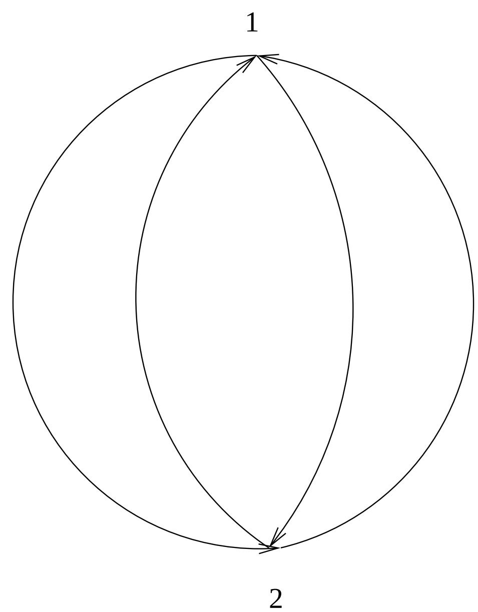

Figure 10: The configuration space of a parellelogram which is not a square