MATH 321 Manifolds and Differential Forms (II)

Homework 3 Solution

Due September 20, 3:00 p.m.

The is the graph of the parametrized curve in Exercise 3.14, presented in the recitation of 09/26. Note as $t \to +\infty$, the curve goes infinitely close to the origin point, but never actually returns. As $t \to -1$, the curve goes infinitely close to the straight line, but never contacts.

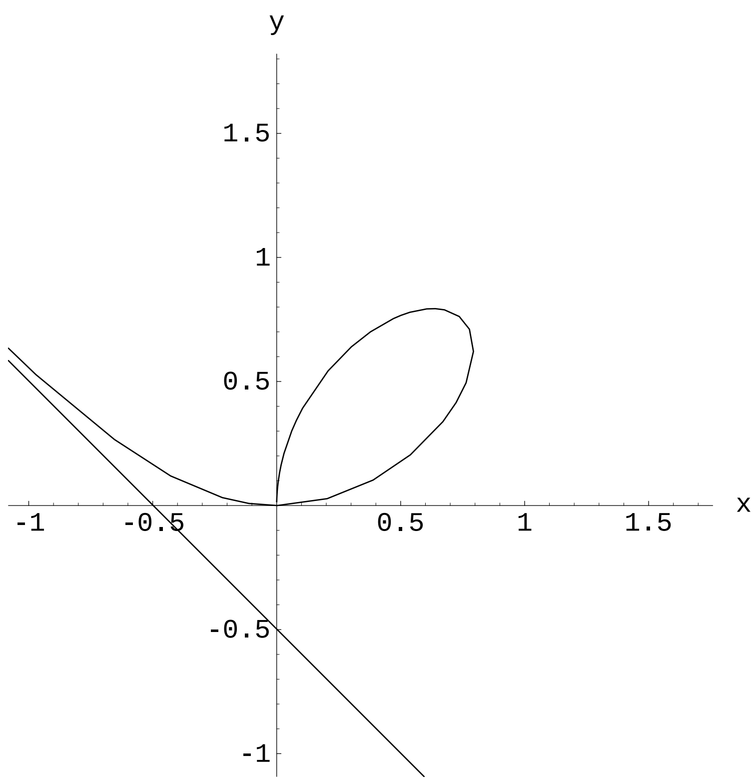

Figure 1: Graph of the curve $f(t) = \frac{1}{2}(e^t + e^{-t}, e^t - e^{-t})$

2.10 (4 points)

(i) Proof: Working on the recursive definitions, we get $P_1(r) = r$, $P_2(r, \theta_1) = (r\cos\theta_1, r\sin\theta_1)$, and $P_3(r, \theta_1, \theta_2) = (r\cos\theta_1\cos\theta_2, r\sin\theta_1\cos\theta_2, r\sin\theta_2)$. $\square$

(ii) Solution:

$$P_4(r,\theta_1,\theta_2,\theta_3) = (r\cos\theta_1\cos\theta_2\cos\theta_3, r\sin\theta_1\cos\theta_2\cos\theta_3, r\sin\theta_2\cos\theta_3, r\sin\theta_3)$$

□

(iii) Proof: Let P_k^i be the i-th entry of the column vector P_k, then $P_{n+1}^i = P_n^i \cos\theta_n$, for $1 \le i \le n$. So

$$dP_{n+1} = \begin{pmatrix} \frac{\partial P_{n+1}^1}{\partial r} & \dots & \frac{\partial P_{n+1}^1}{\partial \theta_{n-1}} & \frac{\partial P_{n+1}^1}{\partial \theta_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_{n+1}^n}{\partial r} & \dots & \frac{\partial P_{n+1}^n}{\partial \theta_{n-1}} & \frac{\partial P_{n+1}^n}{\partial \theta_n} \\ \frac{\partial P_{n+1}^{n+1}}{\partial r} & \dots & \frac{\partial P_{n+1}^{n+1}}{\partial \theta_{n-1}} & \frac{\partial P_{n+1}^{n+1}}{\partial \theta_n} \end{pmatrix} = \begin{pmatrix} \cos\theta dP_n & -sin\theta_n P_n \\ v & r\cos\theta_n \end{pmatrix}$$

where $v = (\sin\theta_n, 0, 0, \dots, 0)$. □

(iv) Proof: $\det dP_{n+1} = r\cos\theta_n \det(\cos\theta_n dP_n) + (-1)^{n+2}\sin\theta_n \det B$, where

$$B = \begin{pmatrix} \frac{\partial P_{n+1}^1}{\partial \theta_1} & \dots & \frac{\partial P_{n+1}^1}{\partial \theta_{n-1}} & \frac{\partial P_{n+1}^1}{\partial \theta_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_{n+1}^n}{\partial \theta_1} & \dots & \frac{\partial P_{n+1}^n}{\partial \theta_{n-1}} & \frac{\partial P_{n+1}^n}{\partial \theta_n} \end{pmatrix}$$

Note

$$\frac{\partial}{\partial\theta_n}\begin{pmatrix} P_{n+1}^1 \\ \vdots \\ P_{n+1}^n \end{pmatrix} = \frac{\partial}{\partial\theta_n}\begin{pmatrix} \cos\theta_n P_n^1 \\ \vdots \\ \cos\theta_n P_n^n \end{pmatrix} = -\sin\theta_n \begin{pmatrix} P_n^1 \\ \vdots \\ P_n^n \end{pmatrix}$$

where the second equality is due to $P_{n+1}^i = P_n^i\cos\theta_n$, for $1 \le i \le n$. So

$$detB = -\sin\theta_n \cos^{n-1}\theta_n det \begin{pmatrix} \frac{\partial P_n^1}{\partial \theta_1} & \dots & \frac{\partial P_n^1}{\partial \theta_{n-1}} & P_n^1 \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_n^n}{\partial \theta_1} & \dots & \frac{\partial P_n^n}{\partial \theta_{n-1}} & P_n^n \end{pmatrix}$$

As $P_n/r = \partial P_n/\partial r$, we conclude

$$detB = -r\sin\theta_n \cos^{n-1}\theta_n det \begin{pmatrix} \frac{\partial P_n^1}{\partial \theta_1} & \dots & \frac{\partial P_n^1}{\partial \theta_{n-1}} & \frac{P_n^1}{\partial r} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_n^n}{\partial \theta_1} & \dots & \frac{\partial P_n^n}{\partial \theta_{n-1}} & \frac{P_n^n}{\partial r} \end{pmatrix}$$

Hence $\det dP_{n+1} = r\cos^{n+1}\theta_n detdP_n + r\sin^2\theta_n\cos^{n-1}\theta_n detdP_n = r\cos^{n-1}\theta_n detdP_n$ □

(v) Solution: $\det dP_1 = 1$, $\det dP_2 = r$, $\det dP_3 = r\cos\theta_2 \det P_2 = r^2\cos\theta_2$.

□

(vi) Solution: $\det dP_n = r^{n-1}\cos^{n-2}\theta_{n-1}\cos^{n-3}\theta_{n-2}\dots\cos\theta_2$. □

3.1. (4 points) Proof: P is obviously C^∞, and the Jacobian matrix of P is

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

which has a nonzero determinant r. So P is also one-to-one. $P^{-1}(x,y) = (\sqrt{x^2+y^2}, ctan^{-1}(x/y))$ is easily seen to be C^∞, on V. So P is a diffeomorphism from U to V. $\square$

3.3. (3 points) Proof: If f is a diffeomorphism, then f is one-to-one, which implies equation $A\mathbf{x} = 0$ has only zero solution. This implies $\det A \neq 0$.

Conversely, if $\det A \neq 0$, then $A\mathbf{x} = 0$ has only zero solution. So f is one-to-one. As $df(\mathbf{x}) = A$, $\det A \neq 0$ also implies $df(\mathbf{x})$ is one-to-one. Finally, $f^{-1}(\mathbf{y}) = A^{-1}(\mathbf{y} - b)$. So f^{-1} is one-to-one and C^∞. $\square$

3.4. (4 points) Proof: $df(\mathbf{x})$ is not of rank 1 if and only if $df(\mathbf{x}) = \mathbf{0}$. If c is a singular value of f, then there exists $\mathbf{x_0}$, such that $f(\mathbf{x_0}) = c$ and $df(\mathbf{x_0}) = \mathbf{0}$. This implies by Exercise 2.4 that $kf(\mathbf{x_0}) = 0$, i.e. $c = 0$. So 0 is the only POSSIBLE singular value of f. However, it's not true that 0 is necessarily a singular value. For example, $f(x,y) = x + y$ is homogeneous of degree 1, but $df(x,y) = (1,1)$ never vanishes. $\square$

3.5. (6 points) Proof: $df(\mathbf{x}) = (2a_1x_1, 2a_2x_2, \ldots, 2a_nx_n)$. So $df(\mathbf{x}) = \mathbf{0}$ is not of rank 1 if and only if $\mathbf{x} = \mathbf{0}$. So the only singular value $c = f(\mathbf{0}) = 0$. As to the graphs of $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = c$ $(c \neq 0)$, we can WLOG assume $c = 1$ and discuss the following typical cases:

Note in Figure 3 and 4, the upper part and the lower part atucally intersect, and their intesection satisfies the equations: $x_3 = 0$, $1 + a_1x_1^2 = a_2x_2^2$. Meanwhile, in Figure 5 and 6, the lowest point of the upper part has height 2, and the hightest point of the lower part has height -2. So these two parts don't conta

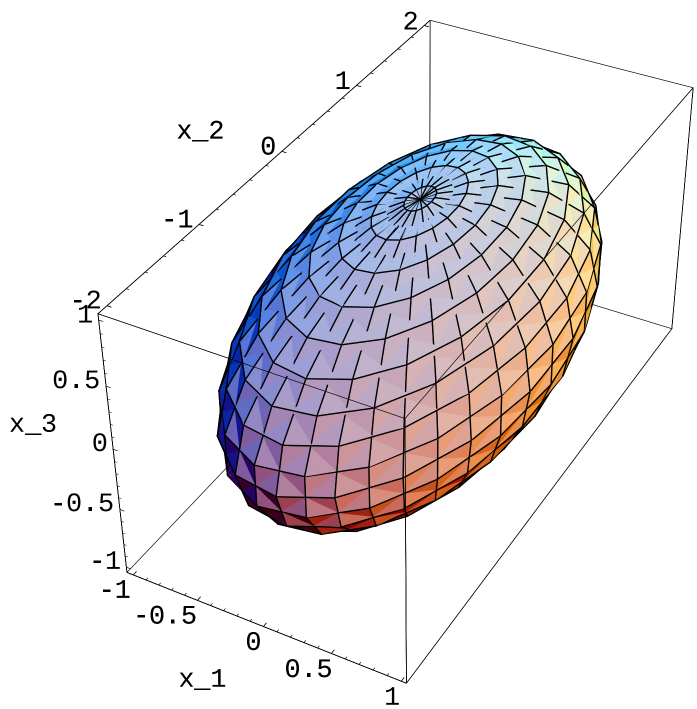

Figure 2: $a_1 > 0, a_2 > 0, a_3 > 0$

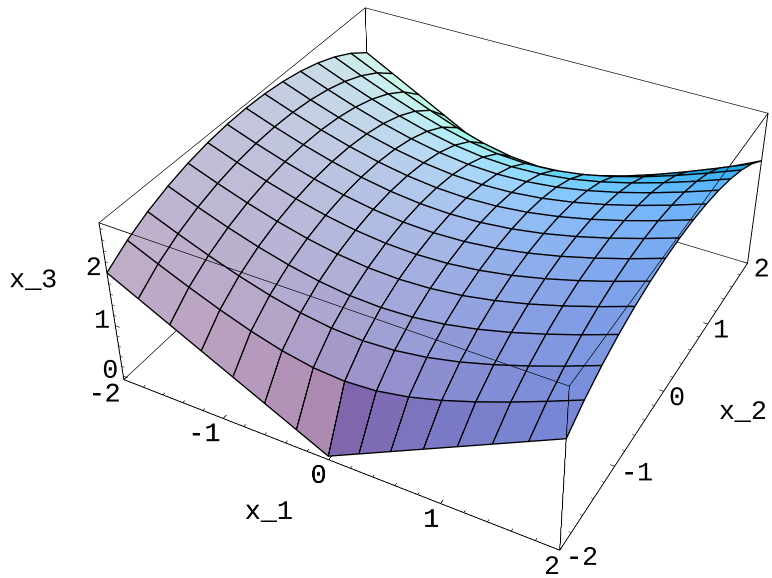

Figure 3: $a_1 < 0, a_2 > 0, a_3 > 0$, the upper part

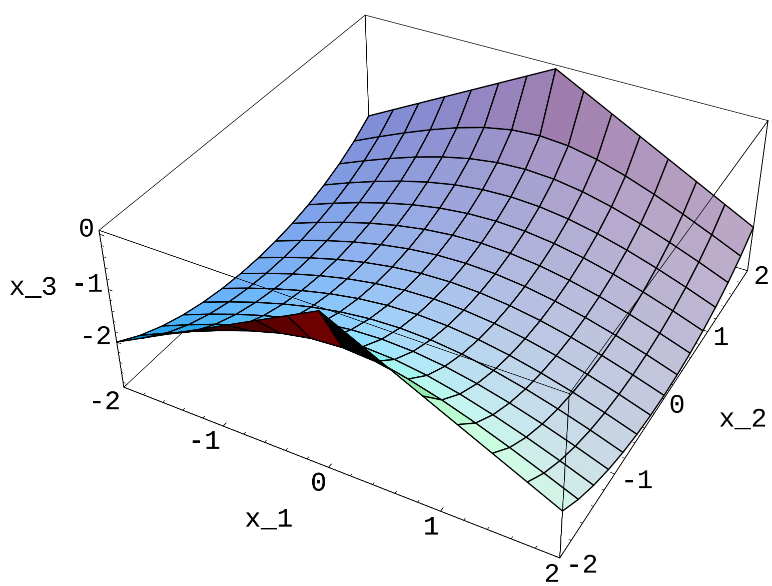

Figure 4: $a_1 < 0, a_2 > 0, a_3 > 0$, the lower part

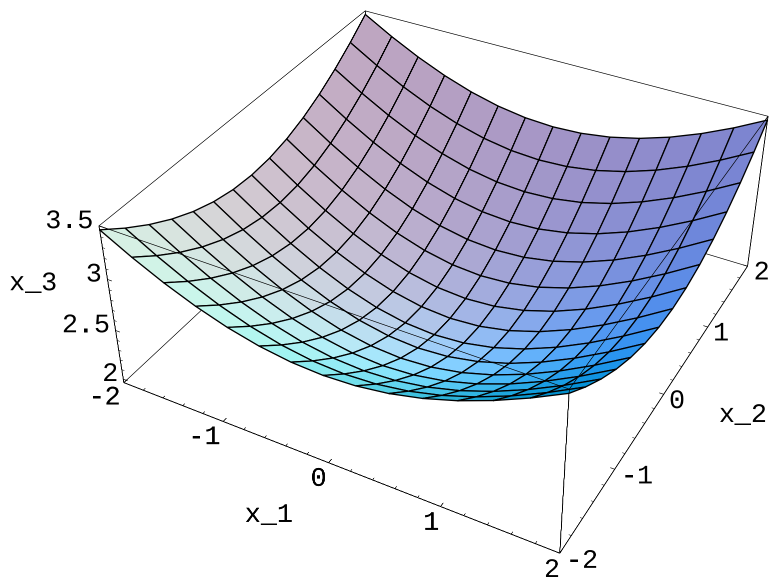

Figure 5: $a_1 < 0, a_2 < 0, a_3 > 0$, the upper part

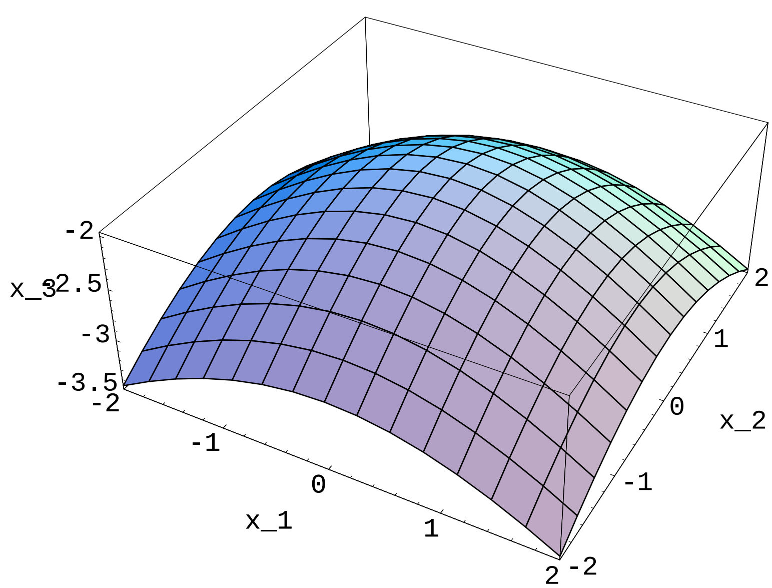

Figure 6: $a_1 < 0, a_2 < 0, a_3 > 0$, the lower part