# MATH 321 Manifolds and Differential Forms (II)

## Homework 4 Solution

Due September 27, 3:00 p.m.

3.8. (4 points) Proof: By definition, $\mathbf{x} \in M$ is critical for $g|_M$ if and only if $dg(\mathbf{x})\mathbf{v} = 0$ for all $\mathbf{v} \in T_{\mathbf{x}}M$, i.e. $dg(\mathbf{x})\mathbf{v} = 0$ if $df(\mathbf{x})\mathbf{v} = 0$ as $T_{\mathbf{x}}M = \ker df(\mathbf{x})$. Note $df(\mathbf{x}) = (gradf_1(\mathbf{x}), \dots, gradf_l(\mathbf{x}))$, so this is equivalent to $dg(\mathbf{x}) \perp \mathbf{v}$ whenever $\mathbf{v} \perp \text{Span}\{gradf_1(\mathbf{x}), \dots, gradf_l(\mathbf{x})\}$. Hence $gradg(\mathbf{x}) = dg(\mathbf{x}) \in \text{Span}\{gradf_1(\mathbf{x}), \dots, gradf_l(\mathbf{x})\}$. □

3.10. (6 points) (i) Solution:

$$g(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x} = \sum_{i=1}^{n} x_i (A\mathbf{x})_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} x_j$$

where $A = (a_{ij})$. So

$$\frac{\partial}{\partial x_k} g(\mathbf{x}) = \sum_{j=1}^{n} a_{kj} x_j + \sum_{i=1}^{n} a_{ik} x_i = 2 \sum_{l=1}^{n} a_{kl} x_l$$

as $A = A^T$. Since $\frac{\partial}{\partial x_k} g(\mathbf{x}) = 2(A\mathbf{x})_k$, we get $gradg(\mathbf{x}) = 2A\mathbf{x}$. □

(ii) Proof: By definition, $\mathbf{x} \in M$ is a critical point of $g|M$ if and only if $dg(\mathbf{x})\mathbf{v} = 0$ for all $\mathbf{v} \in T_{\mathbf{x}}M$, i.e. $dg(\mathbf{x}) \perp T_{\mathbf{x}}M$. Since $M$ is a sphere, $(T_{\mathbf{x}}M)^{\perp} = \text{Span}\{\mathbf{x}\}$. Hence $\mathbf{x} \in M$ is a critical point of $g|_M$ if and only if $dg(\mathbf{x}) = \lambda \mathbf{x}$ for some $\lambda \in \mathbb{R}$. By (i), $dg(\mathbf{x}) = 2A\mathbf{x}$. So $A\mathbf{x} = \lambda \mathbf{x}/2$. This shows $\mathbf{x}$ is an eigenvector for $A$. Obviously $||\mathbf{x}|| = 1$. □

(iii) Proof:

$$g(\mathbf{x})\mathbf{x} = (\mathbf{x} \cdot A\mathbf{x})\mathbf{x} = \mathbf{x}(A\mathbf{x})^T \mathbf{x} = \mathbf{x}\mathbf{x}^T A^T \mathbf{x} = A\mathbf{x}$$

by $A = A^T$ and $||\mathbf{x}|| = 1$. So $g(\mathbf{x})$ is the corresponding eigenvalue of $\mathbf{x}$. □

3.12. (5 points) Proof: $(t - a \sin t)' = 1 - a \cos t$. Since $a \in (0, 1)$, $1 - a \cos t > 0$. So $t - a \sin t$ is increasing. This implies $f$ is i-1.

$df(t) = (1 - a \cos t, a \sin t)$. So $df(t)$ has rank 0 if and only if $a \cos t = 1$ and $\sin t = 0$. This is impossible since $|a \cos t| \leq a < 1$.

To see $f^{-1}$ is continuous, it's beneficial to look at the graph of $f$. Then we can see from the graph that $f^{-1}$ is continuous. (The graph is on the next page.) □

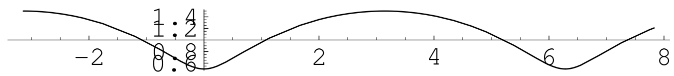


Figure 1: Graph of the curve $f(t) = (t - a\sin t, 1 - a\cos t)$

4.1. (5 points)
(i) Solution: $d(e^{xyz}dx) = -xze^{xyz}dxdy - xye^{xyz}dxdz$. □
(ii) Solution:

$$\begin{aligned} d(\sum_{i=1}^{n} x_i^2 dx_1 \ldots \hat{dx_i} \ldots dx_n) &= \sum_{i=1}^{n} 2x_i dx_i dx_1 \ldots \hat{dx_i} \ldots dx_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} 2x_i dx_1 dx_2 \ldots dx_n \end{aligned}$$

□

(iii) Solution:

$$\begin{aligned} & d(||\mathbf{x}||^p \sum_{i=1}^{n} (-1)^{i+1} x_i dx_1 \ldots \hat{dx_i} \ldots dx_n) \\ &= \sum_{i=1}^{n} (-1)^{i+1} d(x_i ||\mathbf{x}||^p) dx_1 \ldots \hat{dx_i} \ldots dx_n \\ &= \sum_{i=1}^{n} (-1)^{i+1} (||\mathbf{x}||^p + \frac{p}{2} x_i ||\mathbf{x}||^{p-2} 2x_i) dx_i dx_1 \ldots \hat{dx_i} \ldots dx_n) \\ &= \sum_{i=1}^{n} (||\mathbf{x}||^p + p x_i^2 ||\mathbf{x}||^{p-2}) dx_1 \ldots dx_i \ldots dx_n \\ &= (n+p) ||\mathbf{x}||^p dx_1 \ldots dx_i \ldots dx_n \end{aligned}$$

So this form is closed if and ony if $p = -n$. □