MATH 321 Manifolds and Differential Forms (II)
Homework 5 Solution

Due October 11, 3:00 p.m.

4.6 (5 points) Proof: We only prove the case $n = 3$.

$$g(\mathbf{x}) = \int_0^{x_1} f_1(t, x_2, x_3)dt + \int_0^{x_2} f_2(0, t, x_3)dt + \int_0^{x_3} f_3(0, 0, t)dt$$

So

$$\begin{aligned} dg &= f_1(x_1, x_2, x_3)dx_1 + \left[\int_0^{x_1} \frac{\partial f_1}{\partial x_2}(t, x_2, x_3)dt + f_2(0, x_2, x_3)\right] dx_2 + \\ &\left[\int_0^{x_1} \frac{\partial f_1}{\partial x_3}(t, x_2, x_3)dt + \int_0^{x_2} \frac{\partial f_2}{\partial x_3}(0, t, x_3)dt + f_3(0, 0, x_3)\right] dx_3 \\ &= f_1(\mathbf{x})dx_1 + \left[\int_0^{x_1} \frac{\partial f_2}{\partial x_1}(t, x_2, x_3)dt + f_2(0, x_2, x_3)\right] dx_2 \\ &+ \left[\int_0^{x_1} \frac{\partial f_3}{\partial x_1}(t, x_2, x_3)dt + \int_0^{x_2} \frac{\partial f_3}{\partial x_2}(0, t, x_3)dt + f_3(0, 0, x_3)\right] dx_3 \\ &= f_1(\mathbf{x})dx_1 + f_2(\mathbf{x})dx_2 + f_3(\mathbf{x})dx_3 \\ &= \alpha \end{aligned}$$

The second = is due to α is closed and hence $\partial f_i/\partial x_j = \partial f_j/\partial x_i$. The third = is due to fundamental theorem of calculus. □

4.8 (4 points)

$$\begin{aligned} d[(k+1)g] &= \sum_{i=1}^n d(x_i f_i(\mathbf{x})) = \sum_{i=1}^n \sum_{j=1}^n (\delta_{ij} f_i + x_j \frac{\partial f_i}{\partial x_j})dx_j \\ &= \sum_{i=1}^n \sum_{j=1}^n (\delta_{ij} f_i + x_j \frac{\partial f_j}{\partial x_i})dx_j \\ &= \sum_j f_j dx_j + \sum_j dx_j \sum_i x_i \frac{\partial f_j}{\partial x_i} \\ &= \alpha + \sum_j dx_j (k f_j) \\ &= \alpha + k\alpha \\ &= (k+1)\alpha \end{aligned}$$

The third = comes from the fact α is closed and hence $\partial f_i/\partial x_j = \partial f_j/\partial x_i$. And the fifth = is by Exercise 2.4. □

4.10 (5 points) Proof: Since β is exact, we can find a differential form γ such that $d\gamma = \beta$. Suppose α is a k-form, then

$$d((-1)^k \alpha\gamma) = (-1)^k (d\alpha)\gamma + (-)^k (-1)^k \alpha d\gamma = \alpha\beta$$

So, $\alpha\beta$ is exact. □

4.11 (4 points) Proof: WLOG, we assume $\alpha = dx_I$, i.e. α is a monomial with constant coefficient 1. Here I is an increasing multi-indices of degree k. We let J be the index set complementary to I. We also use the notation ε_I and ε_J as defined in the notes (page 26). Then

$$*\alpha = *(dx_I) = \varepsilon_I dx_J, * * \alpha = \varepsilon_I * (dx_J) = \varepsilon_I \varepsilon_J dx_I$$

We conclude $\varepsilon_I \varepsilon_J = (-1)^{k(n-k)} = (-1)^{kn+k}$ by the following equalities

$$dx = \varepsilon_I dx_I dx_J = \varepsilon_I (-1)^{k(n-k)} dx_J dx_I = \varepsilon_I (-1)^{k(n-k)} \varepsilon_J dx$$

So $* * \alpha = (-1)^{kn+k} dx_I$. □

5.4 (4 points)
(i) Length=$2(j-i)-1$, sign=-1.
(ii) Length=$n(n-1)/2$, sign=$(-1)^{n(n-1)/2}$.

□