MATH 321 Manifolds and Differential Forms (II)

Homework 7 Solution

Due October 26, 3:00 p.m.

5.9 (4 points)

(i) Proof: By 5.8 (ii),

$$\begin{aligned} &\det A \\ =&\det A^T \\ =&\sum_{\sigma\in S_n} sign(\sigma)a_{\sigma(1),1}\dots a_{\sigma(n),n} \\ =&\sum_{\sigma\in S_n,\sigma(1)=1} sign(\sigma)a_{1,1}\dots a_{\sigma(n),n}+\sum_{\sigma\in S_n,\sigma(1)\neq 1} sign(\sigma)a_{\sigma(1),1}\dots a_{\sigma(n),n}\end{aligned}$$

By condition, $a_{\sigma(1),1}=0$ if $\sigma(1)\neq 1$. So

$$\det A = a_{11}\sum_{\tau\in S_{n-1}} sign(\tau)a_{\tau(2),2}\dots a_{\tau(n),n}=a_{11}\det A_{11}$$

The last equality is by 5.8 (ii) again. □

(ii) Proof: We let $A=(\alpha_\mathbf{1},\alpha_\mathbf{2},\dots,\alpha_\mathbf{n})$ and $\mathbf{e_j}=(0,\dots,0,1,0,\dots,0)^T$ where 1 only appears in the j-th slot. Note $\alpha_\mathbf{j}=(a_{1j},\dots,a_{nj})^T=\sum_i a_{ij}\mathbf{e_j}$, we then by the multilinearity of the determinant function,

$$\begin{aligned} &\det A \\ =&\sum_j a_{ij}\det(\alpha_\mathbf{1},\dots,\mathbf{e_j},\dots,\alpha_\mathbf{n}) \\ =&\sum_j a_{ij}(-1)^{i+j}\begin{vmatrix} 1 & a_{i1} & \dots & a_{i,j-1} & a_{i,j+1} & \dots & a_{in} \\ 0 & & & A_{ij} & & & \end{vmatrix} \\ =&\sum_j a_{ij}(-1)^{i+j}\det A_{ij}\end{aligned}$$

The last equality is by part (i). □

5.10 (4 points) Proof: First observe that $\det A$ is a polynomial of degree n-1 and if $x_i = x_j$ where $i \neq j$, then we have $\det A = 0$. So $\det A$ must have facotrs of the form $(x_i - x_j)$ where $i \neq j$. Hence $\det A$ is the product of $\prod_{i \neq j}(x_i - x_j)$ and a polynomial Q. By counting the combinations, we see Q has degree of 0. And finally it's easy to see Q is 1. $\square$

5.11 (4 points) Solution:

$$
\begin{aligned}
P^*dx &= \cos\phi\cos\theta dr - r\sin\phi\cos\theta d\phi - r\cos\phi\sin\theta d\theta \\
P^*dy &= \cos\phi\sin\theta dr - r\sin\phi\sin\theta d\phi + r\cos\phi\cos\theta d\theta \\
P^*dz &= \sin\phi dr + r\cos\phi d\phi \\
P^*(dxdy) &= r\cos^2\phi dr d\theta + r^2\sin\phi\cos\phi d\theta d\phi \\
P^*(dydz) &= r\sin\theta d\phi dr + r\cos\phi\sin\phi\cos\theta d\theta dr + r^2\cos^2\phi\cos\theta d\theta dr \\
P^*(dxdz) &= r\cos\theta dr d\theta - r\sin\phi\cos\phi\sin\theta d\theta dr - r^2\cos^2\phi\sin\theta d\theta d\phi \\
P^*(dxdydz) &= r^2\cos\phi[\cos^2\phi + r\sin^2\phi] dr d\theta d\phi
\end{aligned}
$$

$\square$

6.2 (4 points)

(i) Proof: $d\alpha = (d||\mathbf{x}||^\alpha)\sum x_i dx_i = \alpha||\mathbf{x}||^{\alpha-2}(\sum_j x_j dx_j)(\sum_i x_i dx_i) = \alpha||\mathbf{x}||^{\alpha-2}\sum_{i\neq j} x_i x_j dx_i dx_j = \alpha||\mathbf{x}||^{\alpha-2}\sum_{i<j}(x_i x_j - x_j x_i)dx_i dx_j = 0$. $\square$

(ii) Solution: Let $c(t) = (1-t)\mathbf{x}$ where $t \in [0,1]$. Then $g(\mathbf{x}) = \int_{c_\mathbf{x}} \alpha = \int_0^1 ||(1-t)\mathbf{x}||^\alpha \sum_{i=1}^n -(1-t)x_i^2 dt = -||\mathbf{x}||^{2+\alpha}\int_0^1 (1-t)^{\alpha+1}dt$. So to let $g(\mathbf{x})$ be well-defined, $\alpha + 1 > -1$, i.e. $\alpha > -2$. And in this case, $g(\mathbf{x}) = ||\mathbf{x}||^{2+\alpha}/(\alpha + 2)$. $\square$

(iii) Proof: For $\alpha > -2$, $\int_0^1 (1-t)^{\alpha+1}dt = -1/(\alpha+2)$, and $d||\mathbf{x}||^{2+\alpha} = (\alpha+2)||\mathbf{x}||^\alpha \sum x_i dx_i$. So $dg = \alpha$. $\square$

6.3 (4 points)

(i) Solution: $c(t) = t\mathbf{x}$ where $t \in [1,\infty)$. Then $g(\mathbf{x}) = \int_{c_\mathbf{x}} ||\mathbf{x}||^\alpha \sum_{i=1}^n x_i dx_i = \int_1^\infty t^\alpha ||\mathbf{x}||^\alpha \sum_{i=1}^n t x_i^2 dt = ||\mathbf{x}||^{\alpha+2}\int_1^\infty t^{\alpha+1}dt$. So let g be well-defined, $\alpha + 1 < -1$, i.e. $\alpha < -2$, and hence $g(\mathbf{x}) = -||\mathbf{x}||^{\alpha+2}/(\alpha+2)$. $\square$

(ii) Solution: Tedious computation. (Omitted) $\square$

(iii) Solution: Let $\alpha = -3$, then $\alpha = ||\mathbf{x}||^{-1}\sum_i x_i dx_i$ is Newton's gravitational force, and $g(\mathbf{x}) = 1/||\mathbf{x}||$ is the corresponding potential energy. $\square$