MATH 321 Manifolds and Differential Forms (II)

Homework 8 Solution

Due November 8, 3:00 p.m.

6.5 (4 points) Solution:

(i)

$$\begin{aligned} d\eta &= (x^2+y^2)^{-1/2}dy + y[-1/2(x^2+y^2)^{-3/2}(2xdx+2ydy)] \\ &= (x^2+y^2)^{-1/2}dy - (x^2+y^2)^{-3/2}(xydx+y^2dy) \\ &= \frac{-xydx+x^2dy}{(x^2+y^2)^{3/2}} \end{aligned}$$

By symmetry, $d\xi = \frac{-xydy+y^2dx}{(x^2+y^2)^{3/2}}$. So

$$\begin{aligned} \xi d\eta - \eta d\xi &= \frac{x}{\sqrt{x^2+y^2}}\frac{-xydx+x^2dy}{(x^2+y^2)^{3/2}} - \frac{y}{\sqrt{x^2+y^2}}\frac{-xydy+y^2dx}{(x^2+y^2)^{3/2}} \\ &= \frac{-(x^2y+y^3)dx+(x^3+xy^2)dy}{(x^2+y^2)^2} \\ &= \alpha \end{aligned}$$

□

(ii) Assume there is such a nice θ, then we would have $\cos\theta d\sin\theta - \sin\theta d\cos\theta = d\theta$. Meanwhile, by (6.1) in the notes, LHS=α. So we get $\alpha = d\theta$, which means α is exact. This is a contradiction. □

6.7 (4 points) Solution:

(i)

$$\begin{aligned} W_{\bar{c}} &= \frac{1}{2\pi}\int_0^1 \frac{-c_2(1-t)dc_1(1-t)+c_1(1-t)dc_2(1-t)}{c_1^2(1-t)+c_2^2(1-t)} \\ &= \frac{-1}{2\pi}\int_0^1 \frac{-c_2(1-t)c_1'(1-t)+c_1(1-t)c_2'(1-t)}{c_1^2(1-t)+c_2^2(1-t)}dt \\ &= \frac{1}{2\pi}\int_0^1 \frac{-c_2(s)c_1'(s)+c_1(s)c_2'(s)}{c_1^2(s)+c_2^2(s)}(-ds) \\ &= -W_c \\ &= -k \end{aligned}$$

The geometric intuition is that this new closed curve is still the same curve, except the orientation is reversed. $\square$

(ii)

$$\begin{aligned} W_{\bar{c}} &= \frac{1}{2\pi}\int_0^1 \frac{-\rho c_2 d(c_1\rho) + \rho c_1 d(c_2\rho)}{\rho^2(c_1^2+c_2^2)} \\ &= \frac{1}{2\pi}\int_0^1 \frac{-c_2 dc_1 + c_1 dc_2}{c_1^2+c_2^2} \\ &= W_c \\ &= k \end{aligned}$$

The geometric intuition is that this new curve is the old curve amplified, without crossing with the origin. $\square$

(iii) This is reduced to (ii), by setting $\rho = 1/||c(t)||$. So $W_{\bar{c}} = W_c = k$. $\square$

(iv)

$$W_{\bar{c}} = \frac{1}{2\pi}\int_0^1 \frac{-c_1 dc_2 + c_2 dc_1}{c_1^2+c_2^2} = -W_c = -k$$

The geometric intuition is that the new curve is obtained by reflecting $c(t)$ with respect to the diagonal line of the plane. $\square$

(v) Let $\rho(t) = 1/||c(t)||^2$, then $\bar{c} = \rho(t)(c_1, -c_2)$. So

$$W_{\bar{c}} = \frac{1}{2\pi}\int_0^1 \frac{\rho c_2 d(c_1\rho) + \rho c_1 d(-c_2\rho)}{\rho^2(c_1^2+c_2^2)} = \frac{-1}{2\pi}\int_0^1 \frac{-c_2 dc_1 + c_1 dc_2}{c_1^2+c_2^2} = -W_c = -k$$

The geometric intuition is that the old curve is flipped with respect to the y-axis and meanwhile amplified. $\square$

6.11 (4 points) Proof: We let $g(t) = \int_0^t f(s)ds - \lambda t$, where $\lambda = \int_0^1 f(s)ds$. Then it's easy to check $g(t)$ is the desired function. $\square$

7.7 (4 points) Solution:

(i) We let $\alpha = g$, then $d\alpha = \text{grad} g \cdot d\mathbf{x}$.

(ii) We let $\alpha = f dx + g dy$, then $d\alpha = \frac{\partial f}{\partial y} dy dx + \frac{\partial g}{\partial x} dx dy = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx dy$.

(iii) Let $\alpha = \mathbf{F} \cdot (*d\mathbf{x})$, then $d\alpha = d(\sum_i (-1)^{n-i} F_i dx_1 \cdots d\hat{x}_i \cdots dx_n) = \sum_i \frac{\partial F_i}{\partial x_i} dx_1 \cdots dx_n = \text{div}\mathbf{F} dx$.

(iv) Let $\alpha = \sum_i F_i dx_i$, then it's easy to check $d\alpha = \text{curl}\mathbf{F} \cdot (*d\mathbf{x})$.

$\square$

8.2 (4 points) Proof: First of all, we need to clarify the problem. It's restated as follows: let $(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ be a basis of an n-dimensional vector space V, and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the dual basis of V^*. Let A be an $n \times n$ invertible matrix. Then

$$\begin{pmatrix} \mathbf{v}'_1 \\ \mathbf{v}'_2 \\ \vdots \\ \mathbf{v}'_n \end{pmatrix} = A \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \sum_k a_{1k}\mathbf{v}_k \\ \sum_k a_{2k}\mathbf{v}_k \\ \vdots \\ \sum_k a_{nk}\mathbf{v}_k \end{pmatrix}$$

is a new basis of V. We want to find out the dual basis $(\lambda'_1, \ldots, \lambda'_n)$.

By definition, $\lambda'_i(\mathbf{v}'_j) = \delta_{ij}$. But by the linearity of λ'_i and the above representation of $\mathbf{v}'_j$ in terms of original basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, we have $\lambda'_i(\mathbf{v}'_j) = \sum_k a_{jk}\lambda'_i(\mathbf{v}_k)$. So,

$$(\lambda'_i(\mathbf{v}_1), \ldots, \lambda'_i(\mathbf{v}_n)) \begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{pmatrix} = \delta_{ij}$$

This means, if we let $B = (b_{ij}) = (\lambda'_i(\mathbf{v}_j))$. Then $BA^T = I_{n\times n}$. So $B = (A^T)^{-1} = (A^{-1})^T$. Meanwhile,

$$\lambda'_i = \sum_j \lambda'_i(\mathbf{v}_j)\lambda_j = (\lambda'_i(\mathbf{v}_1), \ldots, \lambda'_i(\mathbf{v}_n)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

So

$$\begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_n \end{pmatrix} = B \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = (A^{-1})^T \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

□