MATH 321 Manifolds and Differential Forms (II)

Homework 9 Solution

Due November 15, 3:00 p.m.

6.4 (2 points) Solution: $g = \log ||\mathbf{x}||$. $\square$

6.8 (6 points) Solution: A good way to do this problem is just observing the graphs of the parameterized curves, probably with the help of Mathematica. The widing numbers are 1, 0, 1 and 2, respetively. The following are the graphs of two of them. $\square$

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Figure 1: Graph of the curve $c(t) = (\cos^3 t, \sin^3 t)$, where $t \in [0, 2\pi)$.

Figure 2: Graph of the curve $c(t) = ((2\cos t + 1)\cos t - 1/2, (2\cos t + 1)\sin t)$, where $t \in [0, 2\pi)$.

8.3 (4 points) Proof: Consider $\mu(u+v, u+v) = 0$. Expand the LHS by bilinearity, we get LHS$= \mu(u, u+v) + \mu(v, u+v) = [\mu(u,u) + \mu(u,v)] + [\mu(v,u) + \mu(v,v)] = \mu(u,v) + \mu(v,u)$. For general case, the condition should be formulated in the following way: $\mu(u_1, u_2, \ldots, u_n) = 0$ if for some $i \neq j$, $u_i = u_j$. Then, by the multilinearity and same reasoning, we can show $\mu(u_1, u_2, \ldots, u_n) = sign(\sigma)\mu(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)})$. For example, let us show $\mu(u_1, u_2, \ldots, u_n) = -\mu(u_2, u_1, \ldots, u_n)$, we only need to expand the LHS of the equation $\mu(u_1 + u_2, u_1 + u_2, u_3, \ldots, u_n) = 0$. $\square$

8.5 (5 points) Proof: Let's assume $\mathbf{x} = (a_1, a_2, a_3)^T$, $\mathbf{y} = (b_1, b_2, b_3)^T$. Then

$$\begin{aligned}
\mathbf{x}^T \wedge \mathbf{y}^T &= (a_1, a_2, a_3) \wedge (b_1, b_2, b_3) \\
&= (\sum_i a_i dx_i) \wedge (\sum_j b_j dx_j) \\
&= (a_1 b_2 - a_2 b_1) dx_1 dx_2 + (a_2 b_3 - a_3 b_2) dx_2 dx_3 + (a_1 b_3 - a_3 b_1) dx_1 dx_3 \\
&= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} dx_1 dx_2 + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} dx_2 dx_3 + \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} dx_1 dx_3
\end{aligned}$$

So,

$$*(\mathbf{x}^T \wedge \mathbf{y}^T) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} dx_3 + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} dx_1 + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} dx_2$$

This corresponds exactly to $\mathbf{x} \times \mathbf{y}$. □

8.9 (3 points) Proof:

$$\begin{aligned}
L^*(\lambda\mu)(v_1, v_2) &= \lambda\mu(Lv_1, Lv_2) \\
&= det \begin{vmatrix} \lambda(Lv_1) & \lambda(Lv_2) \\ \mu(Lv_1) & \mu(Lv_2) \end{vmatrix} \\
&= det \begin{vmatrix} L^*\lambda(v_1) & L^*\lambda(v_2) \\ L^*\mu(Lv_1) & L^*\mu(Lv_2) \end{vmatrix} \\
&= L^*\lambda L^*\mu(v_1, v_2)
\end{aligned}$$

□