CHAPTER 11

Applications to topology

1. Homotopy

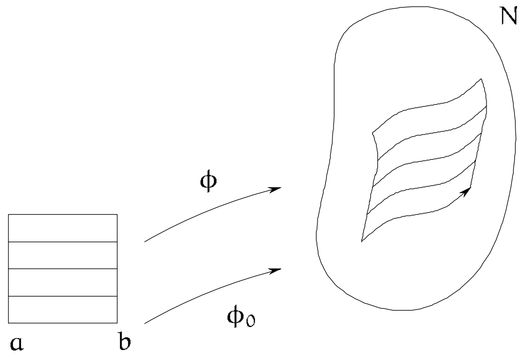
Definition and first examples. Suppose that ϕ_0 and ϕ_1 are two maps from a manifold M to a manifold N and that α is a form on N. What is the relationship between the pullbacks $\phi_0^*\alpha$ and $\phi_1^*\alpha$? There is a reasonable answer to this question if ϕ_0 can be smoothly deformed into ϕ_1. More formally, we say that ϕ_0 and ϕ_1 are *homotopic* if there exists a smooth map $\phi\colon M \times [0,1] \to N$ such that $\phi(\mathbf{x},0) = \phi_0(\mathbf{x})$ and $\phi(\mathbf{x},1) = \phi_1(\mathbf{x})$ for all $\mathbf{x}$ in M. The map ϕ is called a *homotopy*. Instead of $\phi(\mathbf{x},t)$ we often write $\phi_t(\mathbf{x})$. Then each ϕ_t is a map from M to N and we can think of ϕ_t as a family of maps that interpolates between ϕ_0 and ϕ_1, or as a one-second "movie" that at time 0 starts at ϕ_0 and at time 1 ends up at ϕ_1.

11.1. EXAMPLE. Let $M = N = \mathbf{R}^n$ and $\phi_0(\mathbf{x}) = \mathbf{x}$ (identity map) and $\phi_1(\mathbf{x}) = \mathbf{0}$ (constant map). Then ϕ_0 and ϕ_1 are homotopic. A homotopy is given by $\phi(\mathbf{x},t) = (1-t)\mathbf{x}$. This homotopy collapses Euclidean space onto the origin by moving each point radially inward. There are other ways to accomplish this. For instance $(1-t)^2\mathbf{x}$ and $(1-t^2)\mathbf{x}$ are two other homotopies between the same maps. We can also interchange ϕ_0 and ϕ_1: if $\phi_0(\mathbf{x}) = \mathbf{0}$ and $\phi_1(\mathbf{x}) = \mathbf{x}$, then we find a homotopy by reversing time (playing the movie backwards), $\phi(\mathbf{x},t) = t\mathbf{x}$.

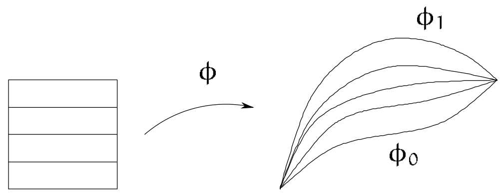
11.2. EXAMPLE. Let $M = N = \mathbf{R}^n - \{\mathbf{0}\}$ and $\phi_0(\mathbf{x}) = \mathbf{x}$ (identity map) and $\phi_1(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ (normalization map). Then ϕ_0 and ϕ_1 are homotopic. A homotopy is given for instance by $\phi(\mathbf{x},t) = \mathbf{x}/\|\mathbf{x}\|^t$ or by $\phi(\mathbf{x},t) = (1-t)\mathbf{x} + t\mathbf{x}/\|\mathbf{x}\|$. Either of these homotopies collapses punctured Euclidean space onto the unit sphere about the origin by smoothly stretching or shrinking each vector until it has length 1.

11.3. EXAMPLE. A manifold M is said to be *contractible* if there exists a point $\mathbf{x}_0$ in M such that the constant map $\phi_0(\mathbf{x}) = \mathbf{x}_0$ is homotopic to the identity map $\phi(\mathbf{x}) = \mathbf{x}$. A specific homotopy $\phi\colon M \times [0,1] \to M$ from ϕ_0 to ϕ_1 is a *contraction* of M onto $\mathbf{x}_0$. (Perhaps "expansion" would be a more accurate term, a "contraction" being the result of replacing t with $1-t$.) Example 11.1 shows that $\mathbf{R}^n$ is contractible onto the origin. (In fact it is contractible onto any point $\mathbf{x}_0$. Can you write a contraction of $\mathbf{R}^n$ onto $\mathbf{x}_0$?) The same formula shows that an open or closed ball around the origin is contractible. We shall see in Theorem 11.15 that punctured n-space $\mathbf{R}^n - \{\mathbf{0}\}$ is *not* contractible.

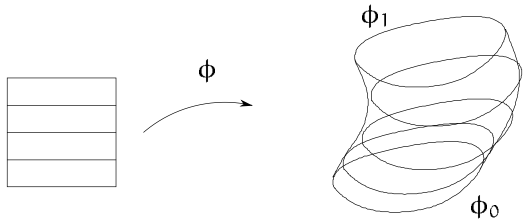
Homotopy of curves. If M is an interval $[a,b]$ and N any manifold, then maps from M to N are nothing but parametrized paths in N. A homotopy of paths can be visualized as a piece of string moving through the manifold N.



A homotopy of paths ϕ in N has *fixed endpoints* if $\phi(0, t) = \phi_0(0)$ and $\phi(1, t) = \phi_0(1)$ for all $t \in [0, 1]$.



Lastly, a loop or closed path can be visualized as a thin rubber band sitting in N. If each path in a homotopy ϕ_t is closed ($\phi_t(a) = \phi_t(b)$ for all t), a homotopy from ϕ_0 to ϕ_1 may be pictured as follows.

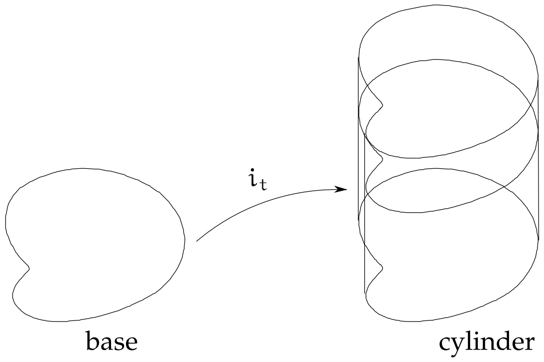


Each loop ϕ_t can also be thought of as a smooth map from the unit circle S^1 into N. Thus the homotopy ϕ can be regarded as a map from the tube $S^1 \times [0, 1]$ to N.

11.4. EXAMPLE. Consider the two circles $\phi_0(\theta) = (\cos\theta, \sin\theta)$ and $\phi_1(\theta) = (2 + \cos\theta, \sin\theta)$ ($0 \leq \theta \leq 2\pi$) in the plane $\mathbf{R}^2$. A homotopy of loops is given by shifting ϕ_0 to the right, $\phi_t(\theta) = (2t + \cos\theta, \sin\theta)$. What if we regard ϕ_0 and ϕ_1 as loops in the punctured plane $\mathbf{R}^2 - \{\mathbf{0}\}$? Clearly the homotopy ϕ does not work, because it moves the path through the forbidden point $\mathbf{0}$. (E.g. $\phi_t(\theta) = \mathbf{0}$ for $\theta = \pi$ and $t = 1/2$.) In fact, however you try to move ϕ_0 to ϕ_1 without opening up the loop you get stuck at the origin, so it seems intuitively clear that there exists no homotopy of loops from ϕ_0 to ϕ_1 in the punctured plane. This is indeed the case, as we shall see in Example 11.9.

The homotopy formula. The product $M \times [0, 1]$ is often called the *cylinder* with *base* M. The two maps defined by $i_0(\mathbf{x}) = (\mathbf{x}, 0)$ and $i_1(\mathbf{x}) = (\mathbf{x}, 1)$ send M to the bottom, resp. the top of the cylinder. A homotopy $i \colon M \times [0, 1] \to M \times [0, 1]$

between these maps is given by the identity map $i(\mathbf{x}, t) = (\mathbf{x}, t)$. ("Slide the bottom to the top at speed 1.")



If M is an open subset of $\mathbf{R}^n$, a $k+1$-form on the cylinder can be written as

$$\alpha = \sum_I f(x,t)dx_I + \sum_J g(x,t)dtdx_J,$$

with I running over multi-indices of length $k+1$ and J over multi-indices of length k. (Here we write the dt in front of the dx's because that is more convenient in what follows.) The *cylinder operator* turns forms on the cylinder into forms on the base lessening the degree by 1,

$$\kappa\colon \Omega^{k+1}(M \times [0,1]) \to \Omega^k(M),$$

by taking the piece of α involving dt and integrating it over the unit interval,

$$\kappa\alpha = \sum_J \left(\int_0^1 g(x,t)dt\right) dx_J.$$

(In particular $\kappa\alpha = 0$ for any α that does not involve dt.) For a general manifold M we can write a $k+1$-form on the cylinder as $\alpha = \beta + dt\gamma$, where β and γ are forms on $M \times [0,1]$ (of degree $k+1$ and k respectively) that do not involve dt. We then define $\kappa\alpha = \int_0^1 \gamma dt$.

The following result will enable us to compare pullbacks of forms under homotopic maps. It can be regarded as an application of Stokes' theorem, but we shall give a direct proof.

11.5. LEMMA (cylinder formula). *Let M be a manifold. Then $\kappa i_1^*\alpha - i_0^*\alpha = d\alpha + d\kappa\alpha$ for all $k+1$-forms α on $M \times [0,1]$. In short,*

$$\boxed{i_1^* - i_0^* = \kappa d + d\kappa.}$$

PROOF. We write out the proof for an open subset of $\mathbf{R}^n$. The proof for arbitrary manifolds is similar. It suffices to consider two cases: $\alpha = f dx_I$ and $\alpha = g dt dx_J$.

Case 1. If $\alpha = f dx_I$, then $\kappa\alpha = 0$ and $d\kappa\alpha = 0$. Also

$$d\alpha = \frac{\partial f}{\partial t} dt dx_I + \sum_i \frac{\partial f}{\partial x_i} dx_i dx_I = \frac{\partial f}{\partial t} dt dx_I + \text{terms not involving } dt,$$

so

$$\begin{aligned} d\kappa\alpha + \kappa d\alpha = \kappa d\alpha &= \left(\int_0^1 \frac{\partial f}{\partial t}(x,t)\,dt\right) dx_I \\ &= \big(f(x,1) - f(x,0)\big) dx_I \\ &= i_1^*\alpha - i_0^*\alpha. \end{aligned}$$

Case 2. If $\alpha = g\,dt\,dx_J$, then $i_0^*\alpha = i_1^*\alpha = 0$ and

$$d\alpha = \sum_i \frac{\partial g}{\partial x_i} dx_i\,dt\,dx_J = -\sum_i \frac{\partial g}{\partial x_i} dt\,dx_i\,dx_J,$$

so

$$\kappa d\alpha = -\sum_{i=1}^n \left(\int_0^1 \frac{\partial g}{\partial x_i}(x,t)\,dt\right) dx_i\,dx_J.$$

Also $\kappa\alpha = \left(\int_0^1 g(x,t)\,dt\right) dx_J$, so

$$\begin{aligned} d\kappa\alpha &= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_0^1 g(x,t)\,dt\right) dx_i\,dx_J \\ &= \sum_{i=1}^n \left(\int_0^1 \frac{\partial g}{\partial x_i}(x,t)\,dt\right) dx_i\,dx_J. \end{aligned}$$

Hence $d\kappa\alpha + \kappa d\alpha = 0 = i_1^*\alpha - i_0^*\alpha$. QED

Now suppose we have a pair of maps ϕ_0 and ϕ_1 going from a manifold M to a manifold N and that $\phi\colon M \times [0,1] \to N$ is a homotopy between ϕ_0 and ϕ_1. For $\mathbf{x}$ in M we have $\phi \circ i_0(\mathbf{x}) = \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x})$, in other words $\phi_0 = \phi \circ i_0$. Similarly $\phi_1 = \phi \circ i_1$. Hence for any $k+1$-form α on N we have $i_0^*\phi^*\alpha = \phi_0^*\alpha$ and $i_1^*\phi^*\alpha = \phi_1^*\alpha$. Applying the cylinder formula to $\phi^*\alpha$ we see that the pullbacks $\phi_0^*\alpha$ and $\phi_1^*\alpha$ are related in the following manner.

11.6. THEOREM (homotopy formula). *Let $\phi_0, \phi_1 : M \to N$ be smooth maps from a manifold M to a manifold N and let $\phi\colon M \times [0,1] \to N$ be a homotopy from ϕ_0 to ϕ_1. Then $\phi_1^*\alpha - \phi_0^*\alpha = \kappa\phi^* d\alpha + d\kappa\phi^*\alpha$ for all $k+1$-forms α on N. In short,*

$$\boxed{\phi_1^* - \phi_0^* = \kappa\phi^* d + d\kappa\phi^*.}$$

In particular, if $d\alpha = 0$ we get $\phi_1^*\alpha = \phi_0^*\alpha + d\kappa\phi^*\alpha$.

11.7. COROLLARY. *If $\phi_0, \phi_1 : M \to N$ are homotopic maps between manifolds and α is a closed form on N, then $\phi_0^*\alpha$ and $\phi_1^*\alpha$ differ by an exact form.*

This implies that $\phi_0^*\alpha$ and $\phi_1^*\alpha$ have the same integral if they are of the correct degree.

11.8. THEOREM. *Let M and N be manifolds and let α be a closed n-form on N, where $n = \dim M$. Suppose M is compact and oriented and has no boundary. Let ϕ_0 and ϕ_1 be homotopic maps from M to N. Then*

$$\int_M \phi_0^*\alpha = \int_M \phi_1^*\alpha.$$

PROOF. By Corollary 11.7, $\phi_1^*\alpha - \phi_0^*\alpha = d\beta$ for an $n-1$-form β on M. Hence by Stokes' theorem

$$\int_M (\phi_1^*\alpha - \phi_0^*\alpha) = \int_M d\beta = \int_{\partial M} \beta = 0,$$

because ∂M is empty. QED

ALTERNATIVE PROOF. Here is a proof based on Stokes' theorem for the manifold with boundary $M \times [0,1]$. The boundary of $M \times [0,1]$ consists of two copies of M, namely $M \times \{1\}$ and $M \times \{0\}$, the first of which is counted with a plus sign and the second with a minus. Therefore, if $\phi\colon M \times [0,1] \to N$ is a homotopy between ϕ_0 and ϕ_1,

$$0 = \int_{M\times[0,1]} \phi^* d\alpha = \int_{M\times[0,1]} d\phi^*\alpha = \int_{\partial(M\times[0,1])} \phi^*\alpha = \int_M \phi_1^*\alpha - \int_M \phi_0^*\alpha.$$

QED

11.9. EXAMPLE. Take $M = S^1$, $N = \mathbf{R}^2 - \{\mathbf{0}\}$ and $\alpha = (-y\,dx + x\,dy)/(x^2+y^2)$. Let $\phi_0(\theta) = (\cos\theta, \sin\theta)$ and $\phi_1(\theta) = (2+\cos\theta, \sin\theta)$ be the circles of Example 11.4. Then $\int_{\phi_i}\alpha$ is 2π times the winding number of ϕ_i around the origin, so $\int_{\phi_0}\alpha = 2\pi$ and $\int_{\phi_1}\alpha = 0$. It now follows from Theorem 11.8 that ϕ_0 is not homotopic to ϕ_1.

2. Closed and exact forms re-examined

The homotopy formula throws light on our old problem of when a closed form is exact, which we looked into briefly in Section 3. The answer turns out to depend on the "shape" of the manifold on which the forms are defined. On some manifolds all closed forms (of positive degree) are exact, on others this is true only in certain degrees. Failure of exactness is typically detected by integrating over a submanifold of the correct dimension and finding a nonzero answer. In a certain sense all obstructions to exactness are of this nature. We shall not attempt to say the last word on these problems, but study a few representative special cases. The matter is explored in **[Fla89]** and at a more advanced level in **[BT82]**.

0-forms. A closed 0-form on a manifold is a smooth function f satisfying $df = 0$. This means that f is constant (on each connected component of M). If this constant is nonzero, then f is not exact (because forms of degree -1 are by definition 0). So a closed 0-form is never exact (unless it is 0) for a rather uninteresting reason.

1-forms and simple connectivity. Let us now consider 1-forms on a manifold M. Theorem 6.5 says that the integral of an exact 1-form along a loop is 0. With a stronger assumption on the loop the same is true for arbitrary *closed* 1-forms. A loop $c\colon S^1 \to M$ is *null-homotopic* if it is homotopic to a constant loop. The integral of a 1-form along a constant loop is 0, so from Theorem 11.8 (where we set the M of the theorem equal to S^1) we get the following.

11.10. PROPOSITION. *Let c be a null-homotopic loop in M. Then $\int_c \alpha = 0$ for all closed forms α on M.*

A manifold is *simply connected* if every loop in it is null-homotopic.

11.11. THEOREM. *All closed* 1*-forms on a simply connected manifold are exact.*

PROOF. Let α be a closed 1-form and c a loop in M. Then c is null-homotopic, so $\int_c \alpha = 0$ by Proposition 11.10. The result now follows from Theorem 6.5. QED

11.12. EXAMPLE. The punctured plane $\mathbf{R}^2 - \{0\}$ is not simply connected, because it possesses a nonexact closed 1-form. (See Example 6.6.) In contrast it can be proved that for $n \geq 3$ the sphere S^{n-1} and punctured n-space $\mathbf{R}^n - \{0\}$ are simply connected. Intuitively, the reason is that in two dimensions a loop that encloses the puncture at the origin cannot be crumpled up to a point without getting stuck at the puncture, whereas in higher dimensions there is enough room to slide any loop away from the puncture and then squeeze it to a point.

The Poincaré lemma. On a contractible manifold *all* closed forms of positive degree are exact.

11.13. THEOREM (Poincaré lemma). *All closed* k*-forms on a contractible manifold are exact for* $k \geq 1$.

PROOF. Let M be a manifold and let $\phi\colon M \times [0,1] \to M$ be a contraction onto a point $\mathbf{x}_0$ in M, i.e. a smooth map satisfying $\phi(\mathbf{x}, 0) = \mathbf{x}_0$ and $\phi(\mathbf{x}, 1) = \mathbf{x}$ for all $\mathbf{x}$. Let α be a closed k-form on M with $k \geq 1$. Then $\phi_1^*\alpha = \alpha$ and $\phi_0^*\alpha = 0$, so putting $\beta = \kappa\phi^*\alpha$ we get

$$\begin{aligned} d\beta &= d\kappa\phi^*\alpha \\ &= \phi_1^*\alpha - \phi_0^*\alpha - \kappa d\phi^*\alpha && \text{by Theorem 11.6} \\ &= \alpha, \end{aligned}$$

because $d\alpha = 0$. Hence $d\beta = \alpha$. QED

The proof provides us with a formula for the "antiderivative" β, which can be made quite explicit in certain cases.

11.14. EXAMPLE. Let M be $\mathbf{R}^n$ and let $\phi(\mathbf{x}, t) = t\mathbf{x}$ be the radial contraction. Let $\alpha = \sum_i g_i \, dx_i$ be a 1-form. Then

$$\phi^*\alpha = \sum_i g_i(t\mathbf{x}) d(tx_i) = \sum_i g_i(t\mathbf{x})(x_i\,dt + t\,dx_i),$$

so

$$\beta = \kappa\phi^*\alpha = \sum_i x_i \int_0^1 g_i(t\mathbf{x})\,dt.$$

According to the proof of the Poincaré lemma, the function β satisfies $d\beta = \alpha$ provided that $d\alpha = 0$. It is instructive to compare β with the function f constructed in the proof of Theorem 6.5. (See Exercise 11.6.)

Another typical application of the Poincaré lemma is showing that a manifold is not contractible by exhibiting a closed form that is not exact. For example, the punctured plane $\mathbf{R}^2 - \{\mathbf{0}\}$ is not contractible because it possesses a nonexact closed 1-form. (See Example 6.6.) More generally consider the $n-1$-form

$$\alpha = \frac{\mathbf{x} \cdot (*d\mathbf{x})}{\|\mathbf{x}\|^n}$$

on punctured n-space $\mathbf{R}^n - \{\mathbf{0}\}$.

11.15. THEOREM. *α is a closed but not exact $n-1$-form on punctured n-space. Hence punctured n-space is not contractible.*

PROOF. $d\alpha = 0$ follows from Exercise 4.1(ii). The $n-1$-sphere $M = S^{n-1}$ has unit normal $\mathbf{x}$, so by Corollary 9.11 on M we have $\alpha = \mu$, the volume form. Hence $\int_M \alpha = \operatorname{vol} M \neq 0$. On the other hand, suppose α was exact, $\alpha = d\beta$ for an $n-1$-form β. Then

$$\int_M \alpha = \int_M d\beta = \int_{\partial M} \alpha = 0$$

by Stokes' theorem, Theorem 10.8. This is a contradiction, so α is not exact. It now follows from the Poincaré lemma, Theorem 11.13, that $\mathbf{R}^n - \{\mathbf{0}\}$ is not contractible. QED

Using the same form α, but restricting it to the unit sphere S^{n-1}, we see that S^{n-1} is not contractible. But how about forms of degree not equal to $n-1$? Without proof we state the following fact.

11.16. THEOREM. *On $\mathbf{R}^n - \{\mathbf{0}\}$ and on S^{n-1} every closed form of degree $k \neq 1$, $n-1$ is exact.*

For a compact oriented hypersurface without boundary M contained in in $\mathbf{R}^n - \{\mathbf{0}\}$ the integral

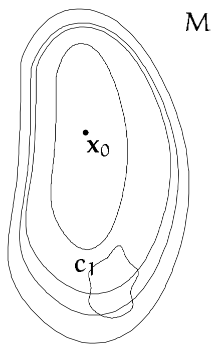
$$\frac{1}{\operatorname{vol}_{n-1} S^{n-1}} \int_M \frac{\mathbf{x} \cdot (*d\mathbf{x})}{\|\mathbf{x}\|^n}$$

is the *winding number* of M about the origin. It generalizes the winding number of a closed curve in $\mathbf{R}^2 - \{\mathbf{0}\}$ around the origin. It can be shown that the winding number in any dimension is always an integer. It provides a measure of how many times the hypersurface wraps around the origin. For instance, the proof of Theorem 11.15 shows that the winding number of the $n-1$-sphere about the origin is 1.

Contractibility versus simple connectivity. Theorems 11.11 and 11.13 suggest that the notions of contractibility and simple connectivity are not independent.

11.17. LEMMA. *A contractible manifold is simply connected.*

PROOF. Use a contraction to collapse any closed curve onto a point.



Formally, let $c_1 : [0,1] \to M$ be a closed curve, $\phi: M \times [0,1] \to M$ a contraction onto $\mathbf{x}_0$. Put $c(s,t) = \phi(c_1(s), t)$. Then c is a homotopy between c_1 and the constant curve $c_0(t) = \phi(c_1(s), 0) = \mathbf{x}_0$ positioned at $\mathbf{x}_0$. QED

The sphere S^{n-1} and punctured n-space $\mathbf{R}^n - \{0\}$ are simply connected for $n \geq 3$ (cf. Example 11.12), although they are not contractible by Theorem 11.15. Thus simple connectivity is weaker than contractibility.

Exercises

11.1. Let c be a k-chain in a region M. Its boundary is a linear combination of $k-1$-cubes, $\partial c = \sum_i a_i c_i$.

(i) Prove that $\sum_i a_i = 0$.
(ii) Conclude that there exists no $k+1$-chain b in M satisfying $\partial b = c$.

11.2. Let M be punctured three-space $\mathbf{R}^3 - \{0\}$ and let $c\colon [0, 2\pi] \times [-\pi/2, \pi/2] \to M$ be the 2-cube given by $c(\theta, \phi) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$. Prove that c is closed and draw a picture to illustrate this fact. Prove that c is not a boundary by integrating $\mathbf{x} \cdot (*d\mathbf{x})/\|\mathbf{x}\|^3$ over c.

11.3. Let $\mathbf{x}_0$ be any point in $\mathbf{R}^n$. By analogy with the radial contraction onto the origin, write a formula for radial contraction onto the point $\mathbf{x}_0$. Deduce that any ball centred at $\mathbf{x}_0$ is contractible.

11.4. A subset M of $\mathbf{R}^n$ is *star-shaped* relative to a point $\mathbf{x}_0 \in M$ if for all $\mathbf{x} \in M$ the straight line segment joining $\mathbf{x}_0$ to $\mathbf{x}$ is entirely contained in M. Show that if M is star-shaped relative to $\mathbf{x}_0$, then it is contractible onto $\mathbf{x}_0$. Give an example of a contractible set that is not star-shaped.

11.5. A subset M of $\mathbf{R}^n$ is *convex* if for all $\mathbf{x}$ and $\mathbf{y}$ in M the straight line segment joining $\mathbf{x}$ to $\mathbf{y}$ is entirely contained in M. Prove the following assertions.

(i) M is convex if and only if it is star-shaped relative to each of its points. Give an example of a star-shaped set that is not convex.
(ii) The closed ball $B = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| \leq r\}$ of radius r centred at $\mathbf{x}_0$ is convex.
(iii) Same for the open ball $B = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| < r\}$.

11.6. Let $\mathbf{x} \in \mathbf{R}^n$ and let $c_{\mathbf{x}}$ be the straight line connecting the origin to $\mathbf{x}$. Let β be the function defined in Example 11.14. Show that $\beta(\mathbf{x}) = \int_{c_{\mathbf{x}}} \alpha$.

11.7. Let α be the k-form $f\,dx_I = f\,dx_{i_1}\,dx_{i_2}\cdots dx_{i_k}$ on $\mathbf{R}^n$ and let $\phi\colon \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ be the radial contraction $\phi(\mathbf{x}, t) = t\mathbf{x}$. Verify that

$$\kappa\phi^*\alpha = \sum_{m=1}^{k} (-1)^{m+1} \left(\int_0^1 f(t\mathbf{x}) t^{k-1}\,dt \right) x_{i_m}\,dx_{i_1}\cdots \widehat{dx_{i_m}} \cdots dx_{i_k}.$$

11.8. Let $\alpha = f\,dx\,dy + g\,dz\,dx + h\,dy\,dz$ be a 2-form on $\mathbf{R}^3$ and let $\phi(x,y,z,t) = t(x,y,z)$ be the radial contraction of $\mathbf{R}^3$ onto the origin. Verify that

$$\kappa\phi^*\alpha = \left(\int_0^1 f(tx,ty,tz)t\,dt \right)(x\,dy - y\,dx) + \left(\int_0^1 g(tx,ty,tz)t\,dt \right)(z\,dx - x\,dz) + \left(\int_0^1 h(tx,ty,tz)t\,dt \right)(y\,dz - z\,dy),$$

and check directly that $d\kappa\phi^*\alpha = \alpha$ if $d\alpha = 0$.

11.9. Let M and N be manifolds and ϕ_0, $\phi_1 : M \to N$ homotopic maps. Show that $\int_c \phi_0^*\alpha = \int_c \phi_1^*\alpha$ for all closed k-chains c in M and all closed k-forms α on N.

11.10. Prove that any two maps ϕ_0 and ϕ_1 from M to N are homotopic if M or N is contractible. (First show that every map $M \to N$ is homotopic to a constant map $\phi(\mathbf{x}) = \mathbf{y}_0$.)

11.11. Let $\mathbf{x}_0 = (2, 0)$ and let M be the twice-punctured plane $\mathbf{R}^2 - \{\mathbf{0}, \mathbf{x}_0\}$. Let $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3 \colon [0, 2\pi] \to M$ be the loops defined by $\mathbf{c}_1(t) = (\cos t, \sin t)$, $\mathbf{c}_2(t) = (2 + \cos t, \sin t)$ and $\mathbf{c}_3(t) = (1 + 2\cos t, 2\sin t)$. Show that $\mathbf{c}_1$ and $\mathbf{c}_2$ are not homotopic. (Construct a 1-form α on M such that the integrals $\int_{\mathbf{c}_1} \alpha$, $\int_{\mathbf{c}_2} \alpha$ and $\int_{\mathbf{c}_3} \alpha$ are distinct.)