CHAPTER 3

Manifolds

The definition

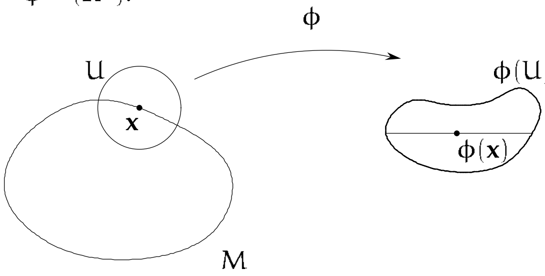
Intuitively, a k-dimensional manifold in $\mathbf{R}^n$ is a subset that in the neighbourhood of every point "looks like" $\mathbf{R}^k$ up to "smooth distortions". Let us now make this notion more precise.

We regard $\mathbf{R}^k$ as the linear subspace of $\mathbf{R}^n$ consisting of all n-vectors $\mathbf{x}$ whose last $n-k$ coordinates are 0: $x_{k+1} = x_{k+2} = \cdots = x_n = 0$.

Let U and V be open subsets of $\mathbf{R}^n$. A *diffeomorphism* from U to V is a map $f\colon U \to V$ which is C^∞ and has a C^∞ inverse $f^{-1}: V \to U$.

A k-*dimensional manifold*[1], or k-*manifold* for short, in $\mathbf{R}^n$ is a subset M such that for each $\mathbf{x} \in M$ there exist an open neighbourhood U of $\mathbf{x}$ in $\mathbf{R}^n$ and a smooth map $\phi\colon U \to \mathbf{R}^n$ satisfying

- ϕ is a diffeomorphism onto an open subset of $\mathbf{R}^n$ (i.e. $\phi(U)$ is open and $\phi\colon U \to \phi(U)$ has a smooth inverse $\phi^{-1}: \phi(U) \to U$);
- $M \cap U = \phi^{-1}(\mathbf{R}^k)$.



The *codimension* of M in $\mathbf{R}^n$ is $n-k$. The *tangent space* to M at $\mathbf{x}$ is

$$T_{\mathbf{x}}M = (d\phi(\mathbf{x}))^{-1}(\mathbf{R}^k),$$

the set of all vectors $\mathbf{v} \in \mathbf{R}^n$ such that $d\phi(\mathbf{x}) \in \mathbf{R}^k$. The elements of $T_{\mathbf{x}}M$ are *tangent vectors* to M at $\mathbf{x}$. (Using the chain rule one can show that $T_{\mathbf{x}}M$ is independent of the choice of the open set U and the map ϕ.)

3.1. EXAMPLE. An open subset V of $\mathbf{R}^n$ is a manifold of codimension 0. Indeed, for any $\mathbf{x} \in V$ we can take $U = V$ and $\phi =$ identity map. The tangent space to V at any point is $\mathbf{R}^n$ itself.

3.2. EXAMPLE. $\mathbf{R}^k$ is a manifold in $\mathbf{R}^n$ of codimension $n-k$. Indeed, for any $\mathbf{x} \in \mathbf{R}^k$ we can simply take $U = \mathbf{R}^n$ and $\phi =$ identity map. Combining this example with the previous one, we see that if V is any open subset of $\mathbf{R}^n$, then $V \cap \mathbf{R}^k$ is a manifold in $\mathbf{R}^n$ of codimension $n-k$. Its tangent space at any point is $\mathbf{R}^k$.

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[1] In the literature this is usually called a *submanifold* of $\mathbf{R}^n$. It is possible to define k-manifolds more abstractly, without reference to an ambient vector space. However, it turns out that practically all abstract manifolds can be embedded into a vector space of sufficiently high dimension. Hence the abstract notion of a manifold is not substantially more general than the notion of a submanifold of a vector space.

Let W be an open subset of $\mathbf{R}^k$ and let $g\colon W \to \mathbf{R}^l$ be a smooth function. The *graph* of g is

$$\operatorname{graph}(g) = \{\, (\mathbf{x}, g(\mathbf{x}) : \mathbf{x} \in W \}.$$

For each $\mathbf{x} \in W$ there is exactly one $\mathbf{y} \in \mathbf{R}^l$ such that $(\mathbf{x}, \mathbf{y})$ is in the graph of g, namely $\mathbf{y} = g(\mathbf{x})$. Thus there is a nice one-to-one correspondence between points in the graph and points in W. It should therefore come as no surprise that, just like W, the graph of g is a k-manifold. However, while W is contained in $\mathbf{R}^k$, the graph is a subset of the product $\mathbf{R}^k \times \mathbf{R}^l$, the set of ordered pairs $(\mathbf{x}, \mathbf{y})$ of vectors $\mathbf{x} \in \mathbf{R}^k$ and $\mathbf{y} \in \mathbf{R}^l$. By stacking the vector $\mathbf{x}$ on top of $\mathbf{y}$ we can think of a pair $(\mathbf{x}, \mathbf{y})$ as a single column vector with $k + l$ entries and thus identify $\mathbf{R}^k \times \mathbf{R}^l$ with $\mathbf{R}^{k+l}$.

3.3. THEOREM. *The graph of a C^∞ function $g\colon W \to \mathbf{R}^l$ is a k-manifold in $\mathbf{R}^{k+l}$.*

PROOF. Put $M = \operatorname{graph}(g)$. Let $U = W \times \mathbf{R}^l$; this is open in $\mathbf{R}^{k+l}$. Note that M is a subset of U. Define the *shear* $\phi\colon U \to U$ by $\phi(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y} - g(\mathbf{x}))$. This is a bijective map. Its inverse is the shear corresponding to the function $-g$: $\phi^{-1}((\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y} + g(\mathbf{x}))$. Both ϕ and its inverse are smooth, so ϕ is a diffeomorphism. Moreover, $\phi^{-1}((\mathbf{x}, \mathbf{y})$ is in $\mathbf{R}^k$ if and only if $\mathbf{y} - g(\mathbf{x}) = \mathbf{0}$, i.e. $\mathbf{y} = g(\mathbf{x})$, or $(\mathbf{x}, \mathbf{y}) \in \operatorname{graph}(g)$. Thus $M = \phi^{-1}(\mathbf{R}^k)$. QED

The definition of a manifold is appealing and intuitive, but difficult to work with in practice. Aside from the above examples, in practice it can be rather hard to decide whether a given subset is a manifold using the definition alone. We need more manageable criteria for a set to be a manifold.

The regular value theorem

Manifolds are often given by equations or systems of equations in the following manner. Let W be an open subset of $\mathbf{R}^n$ and let $f\colon W \to \mathbf{R}^l$ be a smooth map. For $\mathbf{c} \in \mathbf{R}^l$ the set

$$f^{-1}(\mathbf{c}) = \{\mathbf{x} \in U : f(\mathbf{x}) = \mathbf{c}\}$$

is called the *level set* or the *fibre* of f at $\mathbf{c}$. In other words $f^{-1}(\mathbf{c})$ is the set of solutions of the following system of l equations in n variables:

$$\begin{aligned} f_1(x_1, \dots, x_n) &= c_1 \\ f_2(x_1, \dots, x_n) &= c_2 \\ &\vdots \\ f_l(x_1, \dots, x_n) &= c_l \end{aligned}$$

If f is a linear map, these equations are linear. By linear algebra the solution set is then an affine subspace of $\mathbf{R}^n$ of dimension $n - l$, provided that f has rank l (i.e. has l independent columns). We can generalize this idea to nonlinear equations as follows. We say that $\mathbf{c} \in \mathbf{R}^l$ is a *regular value* of f if the Jacobi matrix $df(\mathbf{x})\colon \mathbf{R}^n \to \mathbf{R}^l$ has rank l for all $\mathbf{x} \in f^{-1}(\mathbf{c})$. A vector that is not a regular value is called a *singular* value. (As an extreme, though slightly silly, special case, if $f^{-1}(\mathbf{c})$ is empty, then $\mathbf{c}$ is automatically a regular value.)

The following result is the most useful criterion for a set to be a manifold. It generalizes Theorem 3.3 and is in fact deduced from it by means of the implicit function theorem.

3.4. THEOREM (regular value theorem). *Let W be open in $\mathbf{R}^n$ and let $f\colon W \to \mathbf{R}^l$ be a smooth map. Suppose that $\mathbf{c}$ is a regular value of f and that $M = f^{-1}(\mathbf{c})$ is nonempty. Then M is a manifold in $\mathbf{R}^n$ of codimension l. Its tangent space at $\mathbf{x}$ is the nullspace of* $df(\mathbf{x})$,

$$\boxed{T_{\mathbf{x}}M = \ker df(\mathbf{x}).}$$

PROOF. Let $\mathbf{x} \in M$. Then $df(\mathbf{x})$ has rank l and so has l independent columns. After relabelling the coordinates on $\mathbf{R}^n$ we may assume the last l columns are independent. Let us put $k = n - l$. Identify $\mathbf{R}^n$ with $\mathbf{R}^k \times \mathbf{R}^l$ and correspondingly write an n-vector as a pair $(\mathbf{u}, \mathbf{v})$ with $\mathbf{u}$ a k-vector and $\mathbf{v}$ an l-vector. Also write $\mathbf{x} = (\mathbf{u}_0, \mathbf{v}_0)$. Then the $l \times l$-matrix $(\partial f_i/\partial v_j)_{1 \le i,j \le l}$ is invertible. By the implicit function theorem, Theorem 2.4, there exist open neighbourhoods U of $\mathbf{u}_0$ in $\mathbf{R}^k$ and V of $\mathbf{v}_0$ in $\mathbf{R}^l$ such that for each $\mathbf{u} \in U$ there exists a unique $\mathbf{v} = g(\mathbf{u}) \in V$ satisfying $f(\mathbf{u}, g(\mathbf{u})) = \mathbf{0}$. The function $g\colon U \to V$ is C^∞. In other words $M \cap (U \times V) = \operatorname{graph}(g)$ is the graph of a smooth function. As in the proof of Theorem 3.3 define the shear $\phi\colon U \times V \to U \times \mathbf{R}^l$ by $\phi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v} - g(\mathbf{u}))$. Then $M \cap (U \times V) = \phi^{-1}(\mathbf{R}^k)$. We conclude that M is a k-manifold.

To compute $T_{\mathbf{x}}M$ note that

$$d\phi(\mathbf{x}) = \begin{pmatrix} I_k & 0_{k\times l} \\ -dg(\mathbf{u}_0) & I_l \end{pmatrix},$$

where I_r denotes the $r \times r$-identity matrix and $0_{k\times l}$ the $k \times l$-zero matrix. Let $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^k \times \mathbf{R}^l = \mathbf{R}^n$. Then

$$d\phi(\mathbf{x}) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ -dg(\mathbf{u}_0)\mathbf{a} + \mathbf{b} \end{pmatrix},$$

so $d\phi(\mathbf{x})\left(\begin{smallmatrix} \mathbf{a} \\ \mathbf{b} \end{smallmatrix}\right) \in \mathbf{R}^k$ if and only if $-dg(\mathbf{u}_0)\mathbf{a}+\mathbf{b} = \mathbf{0}$, i.e. $\mathbf{b} = dg(\mathbf{u}_0)\mathbf{a}$. Now $dg(\mathbf{u}_0) = -d_{\mathbf{v}}f(\mathbf{x})^{-1} d_{\mathbf{u}}f(\mathbf{x})$ by the implicit function theorem, so $(\mathbf{a}, \mathbf{b}) \in T_{\mathbf{x}}M$ if and only if $d_{\mathbf{v}}f(\mathbf{x})^{-1} d_{\mathbf{u}}f(\mathbf{x})\mathbf{a} + \mathbf{b} = \mathbf{0}$, in other words $d_{\mathbf{u}}f(\mathbf{x})\mathbf{a} + d_{\mathbf{v}}f(\mathbf{x})\mathbf{b} = \mathbf{0}$. But this is equivalent to $df(\mathbf{x})\left(\begin{smallmatrix} \mathbf{a} \\ \mathbf{b} \end{smallmatrix}\right) = \mathbf{0}$. Hence $T_{\mathbf{x}}M = \ker df(\mathbf{x})$. QED

The case of one single equation ($l = 1$) is especially important. Here df is a single row vector and its transpose is the gradient of f: $df^T = \operatorname{grad} f$. It has rank 1 at $\mathbf{x}$ if and only if it is nonzero, i.e. at least one of the partials of f does not vanish at $\mathbf{x}$. A manifold in $\mathbf{R}^n$ of codimension 1 is commonly called a *hypersurface*. (When $n = 2$ or 3, of course one speaks of a *curve*, resp. a *surface*.) More specifically, the solution set of a scalar equation $f(\mathbf{x}) = c$ is known as a *level hypersurface*.

3.5. COROLLARY (level hypersurfaces). *Let W be open in $\mathbf{R}^n$ and let $f\colon W \to \mathbf{R}$ be a smooth function. Suppose that $M = f^{-1}(c)$ is nonempty and that* $\operatorname{grad} f(\mathbf{x}) \neq \mathbf{0}$ *for all $\mathbf{x}$ in M. Then M is a manifold in $\mathbf{R}^n$ of codimension 1. Its tangent space at $\mathbf{x}$ is the orthogonal complement of* $\operatorname{grad} f(\mathbf{x})$,

$$\boxed{T_{\mathbf{x}}M = (\operatorname{grad} f(\mathbf{x}))^\perp.}$$

3.6. EXAMPLE. Let $W = \mathbf{R}^2$ and $f(x_1, x_2) = x_1x_2$. The equation $f(x_1, x_2) = c$ (where c is a constant) defines a curve $f^{-1}(c) = \{ (x_1, x_2) : x_1x_2 = c \}$ in the plane. Now $\operatorname{grad} f = \left(\begin{smallmatrix} x_2 \\ x_1 \end{smallmatrix}\right)$, so $\operatorname{grad} f$ vanishes only at the origin $\mathbf{0}$, which is contained in $f^{-1}(0)$. This means that any $c \neq 0$ is a regular value of f and therefore $f^{-1}(c)$ is a 1-manifold for $c \neq 0$ by Corollary 3.5. For $c = 0$ the solution set $f^{-1}(c)$ consists of the two coordinate axes and is not a manifold. However, even for $c = 0$ we get a partial result. Let $W' = \mathbf{R}^2 - \{\mathbf{0}\}$ be the punctured plane (origin removed). Then $df(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in W' \cap f^{-1}(\mathbf{c})$ and therefore $W' \cap f^{-1}(\mathbf{c})$ is a 1-manifold. In other words, if we delete the origin from $f^{-1}(\mathbf{c})$ we obtain a 1-manifold.

3.7. EXAMPLE. Let $W = \mathbf{R}^n$ and $f(\mathbf{x}) = \|x\|^2$. Then $\operatorname{grad} f(\mathbf{x}) = 2\mathbf{x}$, so as in the previous example $\operatorname{grad} f$ vanishes only at the origin $\mathbf{0}$, which is contained in $f^{-1}(0)$. So again any $c \neq 0$ is a regular value of f. Clearly $f^{-1}(c)$ is empty for $c < 0$. For $c > 0$, $f^{-1}(c)$ is an $n-1$-manifold, the sphere of radius $\sqrt{c}$ in $\mathbf{R}^n$. Finally, 0 is a singular value of f and $f^{-1}(0) = \{\mathbf{0}\}$ is not an $n-1$-manifold. (It happens to be a 0-manifold, though. So if $\mathbf{c}$ is a singular value, you cannot be certain that $f^{-1}(\mathbf{c})$ is not a manifold. However, even if a singular fibre happens to be a manifold, it is often of the "wrong" dimension.)

We now come to a more sophisticated example of a manifold determined by a large system of equations.

3.8. EXAMPLE. An $n \times n$-matrix A is *orthogonal* if $A^T A = I$. This means that the columns (and also the rows) of A are perpendicular to one another and have length 1. (In other words, they form an ortho*normal* basis of $\mathbf{R}^n$—note the unfortunate inconsistency in the terminology.) The collection of orthogonal matrices form a group under matrix multiplication, which is usually called the *orthogonal group* and denoted by $\mathbf{O}(n)$. Let us prove using the regular value theorem that $\mathbf{O}(n)$ is a manifold. First observe that that $(A^T A)^T = A^T A$, so $A^T A$ is a symmetric matrix. In other words, if $V = \mathbf{R}^{n \times n}$ is the vector space of all $n \times n$-matrices and $W = \{\, C \in V : C = C^T \,\}$ the linear subspace of all symmetric matrices, then

$$f(A) = A^T A$$

defines a map $f\colon V \to W$. Clearly $\mathbf{O}(n) = f^{-1}(I_n)$, so to prove that $\mathbf{O}(n)$ is a manifold it suffices to show that I is a regular value of f. The derivative of f can be computed by the formula

$$df(A)B = \lim_{h \to 0} \frac{1}{h}(f(A + hB) - f(A)). \tag{3.1}$$

Using $f(A+hB) = A^T A + hA^T B + hB^T A + h^2 B^T B$ we find $df_A(B) = BA^T + AB^T$. We need to show that for $A \in \mathbf{O}(n)$ the linear map $df(A)\colon V \to W$ has rank equal to the dimension of W. By linear algebra this amounts to showing that the equation

$$BA^T + AB^T = C \tag{3.2}$$

is solvable for B, given any orthogonal A and any symmetric C. Here is a way of guessing a solution: observe that $C = \frac{1}{2}(C + C^T)$ and first try to solve $BA^T = \frac{1}{2}C$. Left multiplying both sides by A and using $A^T A = I$ gives $B = \frac{1}{2}CA$. It is now easy to check that $B = \frac{1}{2}CA$ is a solution of (3.2).

The embedding theorem

For practical purposes it is often necessary to parametrize a manifold, or a large piece of it. This amounts to representing the manifold as the image, rather than the fibre, of a map. Let us consider therefore a smooth map $f\colon W \to \mathbf{R}^n$, where W is open in $\mathbf{R}^k$. Let $M = f(W)$ be the image of f, i.e. the set of all points

$f(\mathbf{x})$ with $\mathbf{x} \in W$. The hope is that in "good" cases M is a k-dimensional manifold in $\mathbf{R}^n$.

For inspiration let us first investigate the case of a linear map $\mathbf{R}^k \to \mathbf{R}^n$ given by an $n \times k$-matrix A. Here the image $A(\mathbf{R}^k)$ is simply the column space of A, that is the linear subspace of $\mathbf{R}^n$ spanned by its column vectors. Recall the following important fact from linear algebra:

$$\boxed{\text{nullity}(A) + \text{rank}(A) = k.}$$

Here the rank is the number of independent columns of A (in other words the dimension of $A(\mathbf{R}^k)$) and the nullity is the number of independent solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. This equality tells us that the dimension of $A(\mathbf{R}^k)$ is equal to $k - \text{nullity}(A)$. Therefore the image is k-dimensional if and only if the nullity of A is 0. This is the case if and only if A is one-to-one (i.e. $A\mathbf{x}_1 = A\mathbf{x}_2$ implies $\mathbf{x}_1 = \mathbf{x}_2$).

To a certain extent this observation generalizes to the nonlinear case. The map f is called an *embedding* if

- f is one-to-one;
- $df(\mathbf{x})$ is one-to-one for all $\mathbf{x} \in W$;
- if $\mathbf{x}_i$ is a sequence in W and $\lim_{i\to\infty} f(\mathbf{x}_i) = f(\mathbf{x})$ for some $\mathbf{x} \in W$, then $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$.

The last condition is a nuisance, but unfortunately cannot be circumvented. (See Exercise 3.14.) It can be restated by requiring that the inverse of f, considered as a map $f^{-1}\colon f(W) \to W$, should be continuous.

The following result, the proof of which we omit, is deduced from the implicit function theorem in a similar way to the regular value theorem. It is not as user-friendly as the regular value theorem, but nevertheless it has important applications.

3.9. THEOREM (embedding theorem). *Let W be open in $\mathbf{R}^k$ and let $f\colon W \to \mathbf{R}^n$ be an embedding. Then $M = f(W)$ is a k-dimensional manifold in $\mathbf{R}^n$. Its tangent space at $f(\mathbf{x})$ is the column space of* $df(\mathbf{x})$,

$$\boxed{T_{f(\mathbf{x})}M = df(\mathbf{x})(\mathbf{R}^k).}$$

3.10. EXAMPLE. Let $g\colon W \to \mathbf{R}^l$ be a smooth function, with W open in $\mathbf{R}^k$. Let $n = k + l$ and define $f\colon W \to \mathbf{R}^n$ by $f(\mathbf{x}) = (\mathbf{x}, g(\mathbf{x}))$. Note that the image of f is exactly the graph of g. We assert that f is an embedding. Indeed, $(\mathbf{x}_1, g(\mathbf{x}_1)) = (\mathbf{x}_2, g(\mathbf{x}_2))$ implies $\mathbf{x}_1 = \mathbf{x}_2$, so f is one-to-one. Also,

$$df(\mathbf{x}) = \begin{pmatrix} I_k & dg(\mathbf{x}) \end{pmatrix},$$

so $df(\mathbf{x})$ has k independent rows. Therefore $df(\mathbf{x})$ is one-to-one for all $\mathbf{x} \in W$. Finally if $\mathbf{x}_i$ is a sequence in W and $\lim_{i\to\infty}(\mathbf{x}_i, g(\mathbf{x}_i)) = (\mathbf{x}, g(\mathbf{x}))$, then clearly $\mathbf{x}_i \to \mathbf{x}$. Hence f is an embedding. We conclude once again that the graph of g is a k-dimensional manifold in $\mathbf{R}^n$.

It is worthwhile to spell out the one-dimensional case $k = 1$. Let I be an open interval and $f\colon I \to \mathbf{R}^n$ a smooth map. Such a map is called a *(parametrized) path* in $\mathbf{R}^n$. The Jacobi matrix $df(t)$ is a single column vector, called the *velocity vector* and usually denoted by $f'(t)$. Therefore f is an embedding if and only if it is one-to-one, $f'(t) \neq \mathbf{0}$ for all $t \in I$, and $f^{-1}\colon f(I) \to I$ is continuous. If this is the case, then $f(I)$ is called an *embedded path* in $\mathbf{R}^n$.

3.11. COROLLARY (embedded paths). *Let* I *be an open interval and let* $f\colon I \to \mathbf{R}^n$ *be an embedding. Then* $M = f(W)$ *is a* 1*-dimensional manifold in* $\mathbf{R}^n$*. Its tangent space at* $f(t)$ *is the line spanned by the velocity vector* $f'(t)$*,*

$$\boxed{T_{f(t)}M = \{\, af'(t) : a \in \mathbf{R}\,\}.}$$

3.12. EXAMPLE. Consider the path $f\colon \mathbf{R} \to \mathbf{R}^2$ given by $f(\theta) = e^\theta(\cos\theta, \sin\theta)$. Let us check that f is an embedding. Observe first that $\|f(\theta)\| = e^\theta$. Therefore $f(\theta_1) = f(\theta_2)$ implies $e^{\theta_1} = e^{\theta_2}$. The exponential function is one-to-one, so $\theta_1 = \theta_2$. This shows that f is one-to-one. The velocity vector is

$$f'(\theta) = e^\theta \begin{pmatrix} \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix}.$$

Therefore $f'(\theta) = \mathbf{0}$ if and only if $\cos\theta = \sin\theta = 0$, which is impossible because $\cos^2\theta + \sin^2\theta = 1$. So $f'(\theta) \neq \mathbf{0}$ for all θ. Finally suppose (θ_i) is a sequence of numbers such that $\lim_{i\to\infty} f(\theta_i) = f(\theta)$ for some θ. Again using $\|f(\theta)\| = e^\theta$ we get $\lim_{i\to\infty} e^{\theta_i} = e^\theta$. Using the continuity of log we obtain $\lim_{i\to\infty}\theta_i = \theta$. Hence f is an embedding and $f(\mathbf{R})$ is a 1-manifold. The image $f(\mathbf{R})$ is a spiral, which for $t \to -\infty$ converges to the origin and winds infinitely many times around it. Even though $f(\mathbf{R})$ is a manifold, the set $f(\mathbf{R}) \cup \{\mathbf{0}\}$ is not: it has a very nasty singularity at the origin!

Exercises

3.1. Let $U = (-\pi, \pi) \times (0, \infty)$ and $V = \mathbf{R}^2 - \{(x, y) : x \leq 0, y = 0\}$ ($\mathbf{R}^2$ minus a ray). Show that the polar coordinate map $P\binom{r}{\theta} = \binom{r\cos\theta}{r\sin\theta}$ is a diffeomorphism from U to V.

3.2. Show that the function $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is smooth and invertible but does not have a smooth inverse (so it is not a diffeomorphism).

3.3. Let A be an $n \times n$-matrix and $\mathbf{b}$ a fixed n-vector. Define $f\colon \mathbf{R}^n \to \mathbf{R}^n$ by $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. Show that f is a diffeomorphism if and only if $\det A \neq 0$.

3.4. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a homogeneous function of degree k as defined in Exercise 2.4. Show that 0 is the only singular value of f. (Use the result of Exercise 2.4.) Conclude that $f^{-1}(\mathbf{c})$ is an $n-1$-manifold if $\mathbf{c} \neq 0$.

3.5. Let $f(\mathbf{x}) = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$, where the a_i are nonzero constants. Determine the regular and singular values of f. For $n = 3$ draw a picture of the fibre $f^{-1}(\mathbf{c})$ for a regular value $\mathbf{c}$. (You have to distinguish between a few different cases.)

3.6. Compute the dimension of the orthogonal group $\mathbf{O}(n)$ and show that its tangent space at the identity matrix I is the set of all antisymmetric $n \times n$-matrices.

3.7. (i) Let $W = \mathbf{R}^4 - \{\mathbf{0}\}$ and define $f\colon W \to \mathbf{R}$ by $f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$. Show that 0 is a regular value of f.
(ii) Let A be a real 2×2-matrix. Show that $\operatorname{rank} A = 1$ if and only if $\det A = 0$ and $A \neq 0$.
(iii) Let M be the set of 2×2-matrices of rank 1. Show that M is a 3-dimensional manifold.
(iv) Compute T_AM, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

3.8. Let W be an open subset of $\mathbf{R}^n$ and let $f\colon W \to \mathbf{R}^l$ be a smooth map. Let $M = f^{-1}(\mathbf{c})$, where $\mathbf{c}$ is a regular value of f. Let $g\colon W \to \mathbf{R}$ be a smooth function. A point $\mathbf{x} \in M$ is called a *critical point* for the *restricted function* $g|_M$ if $dg(\mathbf{x})\mathbf{v} = 0$ for all tangent vectors $\mathbf{v} \in T_\mathbf{x}M$. Prove that $\mathbf{x} \in M$ is critical for $g|_M$ if and only if there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_l$ such that

$$\operatorname{grad} g(\mathbf{x}) = \lambda_1 \operatorname{grad} f_1(\mathbf{x}) + \lambda_2 \operatorname{grad} f_2(\mathbf{x}) + \cdots + \lambda_l \operatorname{grad} f_l(\mathbf{x}).$$

(Use the characterization of $T_\mathbf{x}M$ given by the regular value theorem.)

3.9. Find the critical points of the function $g(x, y, z) = -x + 2y + 3z$ over the circle C given by

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x + z &= 0. \end{aligned}$$

Where are the maxima and minima of $g|_C$?

3.10. Let $A = A^T$ be a symmetric $n \times n$-matrix and define $g\colon \mathbf{R}^n \to \mathbf{R}$ by $g(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$. Let M be the unit sphere $\{\, \mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| = 1 \,\}$.

(i) Calculate $\operatorname{grad} g(\mathbf{x})$.
(ii) Show that $\mathbf{x} \in M$ is a critical point of $g|_M$ if and only if $\mathbf{x}$ is an eigenvector for A of length 1.
(iii) Given an eigenvector $\mathbf{x}$ of length 1, show that $g(\mathbf{x})$ is the corresponding eigenvalue of $\mathbf{x}$.

3.11. This is a continuation of Exercise 1.3. Define $f\colon \mathbf{R} \to \mathbf{R}^2$ by $f(t) = (t - \sin t, 1 - \cos t)$. Show that f is one-to-one. Determine all t for which $f'(t) = \mathbf{0}$. Is $f(\mathbf{R})$ a manifold at these points?

3.12. Let $a \in (0, 1)$ be a constant. Prove that the map $f\colon \mathbf{R} \to \mathbf{R}^2$ given by $f(t) = (t - a \sin t, 1 - a \cos t)$ is an embedding. (This becomes easier if you first show that $t - a \sin t$ is an increasing function of t.)

3.13. Prove that the map $f\colon \mathbf{R} \to \mathbf{R}^2$ given by $f(t) = \frac{1}{2}(e^t + e^{-t}, e^t - e^{-t})$ is an embedding. Conclude that $M = f(\mathbf{R})$ is a 1-manifold. Compute the tangent space to M at $\binom{1}{1}$ and try to find an equation for M.

3.14. Let I be the open interval $(-1, \infty)$ and let $f\colon I \to \mathbf{R}^2$ be the map $f(t) = (3at/(1 + t^3), 3at^2/(1 + t^3))$. Show that f is one-to-one and that $f'(t) \neq \mathbf{0}$ for all $t \in I$. Is f an embedding and is $f(I)$ a manifold? (Oberve that $f(I)$ is a portion of the curve studied in Exercise 1.4.)