CHAPTER 4

Differential forms on $\mathbf{R}^n$

The notion of a differential form encompasses such ideas as surface and volume forms, the work exerted by a force, the flow of a fluid, and the curvature of a surface, space or hyperspace. An important operation on differential forms is exterior differentiation, which generalizes the operators div, grad and curl of vector calculus. This chapter is an informal introduction to differential forms and their properties.

Elementary properties

A *differential form of degree* k or a k-*form* on $\mathbf{R}^n$ is an expression of the form

$$\alpha = \sum_I f_I \, dx_I .$$

Here I stands for a *multi-index* $(i_1, i_2, \dots, i_k)$ of *degree* k, that is a "vector" consisting of k integer entries ranging between 1 and n. The f_I are smooth functions on $\mathbf{R}^n$ called the *coefficients* of α, and dx_I is an abbreviation for

$$dx_{i_1} dx_{i_2} \cdots dx_{i_k}.$$

(The notation $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ is also often used in the literature to distinguish between this kind of multiplication and another kind, called tensor multiplication.)

If the coefficients f_I are not defined on all of $\mathbf{R}^n$ but on an open subset U of $\mathbf{R}^n$, we say α is a k-form on U. Thus the expression $\log(x^2+y^2)z\,dz$ is not a 1-form on $\mathbf{R}^3$, but on the open set $U = \mathbf{R}^3 - \{\,(x,y,z) : x^2+y^2 \neq 0\,\}$, i.e. outside the z-axis.

You can think of dx_i as an infinitesimal increment in the variable x_i and of dx_I as the volume of an infinitesimal k-dimensional rectangular block with sides $dx_{i_1}, dx_{i_2}, \dots, dx_{i_k}$. (A precise, mathematically satisfactory definition will follow later.) By volume we mean *oriented* volume, which takes into account the order of the variables. Thus, if we interchange two variables, the sign changes:

$$dx_{i_1} dx_{i_2} \cdots dx_{i_q} \cdots dx_{i_p} \cdots dx_{i_k} = -dx_{i_1} dx_{i_2} \cdots dx_{i_p} \cdots dx_{i_q} \cdots dx_{i_k}, \tag{4.1}$$

and so forth. This is called *anticommutativity* or the *alternating property*. In particular, this rule implies $dx_i dx_i = -dx_i dx_i$, so $dx_i dx_i = 0$ for all i.

Let us consider k-forms for some special values of k.

A 0-form on $\mathbf{R}^n$ is simply a smooth function (no dx's).

A general 1-form looks like

$$f_1\,dx_1 + f_2\,dx_2 + \cdots + f_n\,dx_n .$$

A general 2-form has the shape

$$\begin{aligned}\sum_{ij} f_{ij}\,dx_i\,dx_j = {}& f_{11}\,dx_1\,dx_1 + f_{12}\,dx_1\,dx_2 + \cdots + f_{1n}\,dx_1\,dx_n \\ &+ f_{21}\,dx_2\,dx_1 + f_{22}\,dx_2\,dx_2 + \cdots + f_{2n}\,dx_2\,dx_n + \cdots \\ &+ f_{n1}\,dx_n\,dx_1 + f_{n2}\,dx_n\,dx_2 + \cdots + f_{nn}\,dx_n\,dx_n.\end{aligned}$$

Because of the alternating property (4.1) the terms $f_{ii}\,dx_i\,dx_i$ vanish, and a pair of terms such as $f_{12}\,dx_1\,dx_2$ and $f_{21}\,dx_2\,dx_1$ can be grouped together as $(f_{12} - f_{21})\,dx_1\,dx_2$. So we can write 2-forms more economically as

$$\begin{aligned}\sum_{1\le i<j\le n} g_{ij}\,dx_i\,dx_j = {}& g_{12}\,dx_1\,dx_2 + \cdots + g_{1n}\,dx_1\,dx_n \\ &+ g_{23}\,dx_2\,dx_3 + \cdots + g_{2n}\,dx_2\,dx_n + \cdots + g_{n-1,n}\,dx_{n-1}\,dx_n.\end{aligned}$$

Written like this, a 2-form has up to $n + n - 1 + n - 2 + \cdots + 2 + 1 = \frac{1}{2}n(n-1)$ components.

Likewise, a general $n-1$-form can be written as a sum of n components,

$$\begin{aligned}f_1\,dx_2\,dx_3\cdots dx_n + f_2\,dx_1\,dx_3\cdots dx_n + \cdots + f_n\,dx_1\,dx_2\cdots dx_{n-1} \\ = \sum_{i=1}^{n} f_i\,dx_1\,dx_2\cdots\widehat{dx_i}\cdots dx_n,\end{aligned}$$

where $\widehat{dx_i}$ means "drop the factor dx_i".

Every n-form on $\mathbf{R}^n$ can be written as

$$f\,dx_1\,dx_2\cdots dx_n.$$

The special n-form $dx_1\,dx_2\cdots dx_n$ is also known as the *volume form*.

Forms of degree $k > n$ on $\mathbf{R}^n$ are always 0, because at least one variable has to repeat in any expression $dx_{i_1}\cdots dx_{i_k}$. A convention which is sometimes useful is that forms of negative degree are all 0.

In general a form of degree k can be expressed as a sum

$$\alpha = \sum_I f_I\,dx_I,$$

where the I are *increasing* multi-indices, $i_1 < i_2 < \cdots < i_k$. We shall almost always represent forms in this manner. The maximum number of terms occurring in α is then the number of increasing multi-indices of degree k, which is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

("n choose k"). Two k-forms $\alpha = \sum_I f_I\,dx_I$ and $\beta = \sum_I g_I\,dx_I$ (with I ranging over the increasing multi-indices) are equal if and only if $f_I = g_I$ for all I. The space of k-forms on an open set U is denoted by $\Omega^k(U)$.

A form is *constant* if the coefficients f_I are constant functions. The set of constant k-forms is a vector space of dimension $\binom{n}{k}$. A basis of this vector space is given by the forms dx_I, where I ranges over all increasing multi-indices of degree k.

The product of a k-form $\alpha = \sum_I f_I \, dx_I$ and an l-form $\beta = \sum_J g_J \, dx_J$ is defined to be the $k + l$-form

$$\alpha\beta = \sum_{IJ} f_I g_J \, dx_I \, dx_J.$$

Usually many terms in a product cancel out or can be combined. For instance,

$$(y\,dx + x\,dy)(x\,dx\,dz + y\,dy\,dz) = y^2\,dx\,dy\,dz + x^2\,dy\,dx\,dz = (y^2 - x^2)\,dx\,dy\,dz.$$

As an extreme example of such a cancellation, consider an arbitrary form α of degree k. Its pth power α^p is of degree kp, which is greater than n if $k > 0$ and $p > n$. Therefore

$$\alpha^{n+1} = 0$$

for any form on $\mathbf{R}^n$ of positive degree.

The alternating property combines with the multiplication rule to give the following result.

4.1. PROPOSITION (graded commutativity).

$$\boxed{\beta\alpha = (-1)^{kl}\alpha\beta}$$

for all k-forms α and all l-forms β.

PROOF. Let $I = (i_1, i_2, \dots, i_k)$ and $J = (j_1, j_2, \dots, j_l)$. Successively applying the alternating property we get

$$\begin{aligned}
dx_I\,dx_J &= dx_{i_1}\,dx_{i_2}\cdots dx_{i_k}\,dx_{j_1}\,dx_{j_2}\,dx_{j_3}\cdots dx_{j_l} \\
&= (-1)^k\,dx_{j_1}\,dx_{i_1}\,dx_{i_2}\cdots dx_{i_k}\,dx_{j_2}\,dx_{j_3}\cdots dx_{j_l} \\
&= (-1)^{2k}\,dx_{j_1}\,dx_{j_2}\,dx_{i_1}\,dx_{i_2}\cdots dx_{i_k}\,dx_{j_3}\cdots dx_{j_l} \\
&\ \ \vdots \\
&= (-1)^{kl}\,dx_J\,dx_I.
\end{aligned}$$

For general forms $\alpha = \sum_I f_I\,dx_I$ and $\beta = \sum_J g_J\,dx_J$ we get from this $\beta\alpha = \sum_{IJ} g_J f_I\,dx_J\,dx_I = (-1)^{kl}\sum_{IJ} f_I g_J\,dx_I\,dx_J = (-1)^{kl}\alpha\beta$. QED

A noteworthy special case is $\alpha = \beta$. Then we get $\alpha^2 = (-1)^{k^2}\alpha^2$. This equality is vacuous if k is even, but tells us that $\alpha^2 = 0$ if k is odd.

4.2. COROLLARY. *$\alpha^2 = 0$ if α is a form of odd degree.*

The exterior derivative

If f is a 0-form, that is a smooth function, we define df to be the 1-form

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,dx_i.$$

Then we have the *product* or *Leibniz* rule:

$$d(fg) = f\,dg + g\,df.$$

If $\alpha = \sum_I f_I\,dx_I$ is a k-form, each of the coefficients f_I is a smooth function and we define $d\alpha$ to be the $k + 1$-form

$$d\alpha = \sum_I df_I\,dx_I.$$

The operation d is called *exterior differentiation*. An operator of this sort is called a first-order partial differential operator, because it involves the first partial derivatives of the coefficients of a form.

4.3. EXAMPLE. If $\alpha = f dx + g dy$ is a 1-form on $\mathbf{R}^2$, then

$$d\alpha = f_y \, dy \, dx + g_x \, dx \, dy = (g_x - f_y) \, dx \, dy.$$

(Recall that f_y is an alternative notation for $\partial f/\partial y$.) If $\alpha = f dx dy + g dx dz + h dy dz$ is a 2-form on $\mathbf{R}^3$, then

$$d\alpha = f_z dz dx dy + g_y dy dx dz + h_x dx dy dz = (f_z - g_y + h_x) dx dy dz.$$

If α is an n-form on $\mathbf{R}^n$, then $d\alpha$ is of degree $n+1$, so $d\alpha = 0$.

The operator d is linear and satisfies a generalized Leibniz rule.

4.4. PROPOSITION. (i) $d(c_1\alpha_1 + c_2\alpha_2) = c_1 d\alpha_1 + c_2 d\alpha_2$ *for any* k*-forms* α_1 *and* α_2 *and any scalars* c_1 *and* c_2.
(ii) $d(\alpha\beta) = (d\alpha)\beta + (-1)^k \alpha d\beta$ *for any* k*-form* α *and* l*-form* β.

PROOF. The linearity property (i) follows from the linearity of partial differentiation:

$$\frac{\partial(c_1 f_1 + c_2 f_2)}{\partial x_i} = c_1 \frac{\partial f_1}{\partial x_i} + c_2 \frac{\partial f_2}{\partial x_i}$$

for any smooth functions f_1, f_2 and any constants c_1, c_2.

Now let $\alpha = \sum_I f_I dx_I$ and $\beta = \sum_J g_J dx_J$. The Leibniz rule for functions and Proposition 4.1 give

$$\begin{aligned}
d(\alpha\beta) &= \sum_{IJ} d(f_I g_J) dx_I dx_J \\
&= \sum_{IJ} (f_I dg_J + g_J df_I) dx_I dx_J \\
&= \sum_{IJ} \big(df_I dx_I (g_J dx_J) + (-1)^k f_I dx_I (dg_J dx_J)\big) \\
&= (d\alpha)\beta + (-1)^k \alpha d\beta.
\end{aligned}$$

QED

Here is one of the most curious properties of the exterior derivative.

4.5. PROPOSITION. $d(d\alpha) = 0$ *for any form* α. *In short,*

$$\boxed{d^2 = 0.}$$

PROOF. Let $\alpha = \sum_I f_I dx_I$. Then

$$\begin{aligned}
d(d\alpha) &= d\left(\sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i dx_I\right) \\
&= \sum_I \sum_{i,j=1}^n \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j dx_i dx_I \\
&= \sum_I \sum_{1 \le i < j \le n} \left(\frac{\partial^2 f_I}{\partial x_j \partial x_i} - \frac{\partial^2 f_I}{\partial x_i \partial x_j}\right) dx_j dx_i dx_I
\end{aligned}$$

because of the alternating property. Now for any smooth (indeed, C^2) function f the mixed partials $\partial^2 f/\partial x_j \partial x_i$ and $\partial^2 f/\partial x_i \partial x_j$ are equal, so $d(d\alpha) = 0$. QED

Closed and exact forms

A form α is *closed* if $d\alpha = 0$. It is *exact* if $\alpha = d\beta$ for some form β (of degree one less). Proposition 4.5 says that every exact form is closed.

4.6. EXAMPLE. $-y\,dx + x\,dy$ is not closed and therefore cannot be exact. On the other hand $y\,dx + x\,dy$ is closed. It is also exact, because $d(xy) = y\,dx + x\,dy$. For a 0-form (function) f on $\mathbf{R}^n$ to be closed all its partial derivatives must vanish, which means it is constant. A nonzero constant function is not exact, because forms of degree -1 are 0.

Is every closed form of positive degree exact? This problem has interesting ramifications, which we shall explore later. Amazingly the answer depends crucially on the qualitative "shape" of the domain of definition of the form.

Let us consider the simplest case of a 1-form $\alpha = \sum_{i=1}^n f_i\,dx_i$. Determining whether α is exact means to solve the equation $dg = \alpha$ for the function g. This amounts to

$$\frac{\partial g}{\partial x_1} = f_1, \quad \frac{\partial g}{\partial x_2} = f_2, \quad \frac{\partial g}{\partial x_n} = f_n,$$

a system of *first-order partial differential equations*. Finding a solution is sometimes called *integrating* the system. By Proposition 4.5 this is not possible unless α is closed, which amounts to

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

for all $1 \le i < j \le n$. These identities must be satisfied for the system $dg = \alpha$ to be solvable and are therefore called the *integrability conditions*.

4.7. EXAMPLE. Let $\alpha = y\,dx + (z\cos yz + x)\,dy + y\cos yz\,dz$. Then

$$d\alpha = dy\,dx + (z(-y\cos yz) + \cos yz)\,dz\,dy + dx\,dy + (y(-z\cos yz) + \cos yz)\,dy\,dz = 0,$$

so α is closed. Is α exact? Let us solve the equations

$$\frac{\partial g}{\partial x} = y, \quad \frac{\partial g}{\partial y} = z\cos yz + x, \quad \frac{\partial g}{\partial z} = y\cos yz$$

by successive integration. The first equation gives $g = yx + c(y,z)$, where c is a function of y and z only. Substituting into the second equation gives $\partial c/\partial y = z\cos yz$, so $c = \sin yz + k(z)$. Substituting into the third equation gives $k' = 0$, so k is a constant. So $g = yx + \sin yz$ is a solution.

This method works always for a 1-form defined on all of $\mathbf{R}^n$. (See Exercise 4.6.) Hence every closed 1-form on $\mathbf{R}^n$ is exact.

For a 2-form $\alpha = \sum_{1 \le i < j \le n} f_{ij}\,dx_i\,dx_j$ and a 1-form $\beta = \sum_{i=1}^n g_i\,dx_i$ the equation $d\beta = \alpha$ amounts to the system

$$\frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} = f_{ij}.$$

The integrability condition $d\alpha = 0$ comes down to

$$\frac{\partial f_{ij}}{\partial x_k} - \frac{\partial f_{ik}}{\partial x_j} + \frac{\partial f_{jk}}{\partial x_i} = 0$$

for all $1 \leq i < j < k \leq n$. We shall learn how to solve this system, and its higher-degree analogues, later.

The Hodge star operator

Recall that
$$\binom{n}{k} = \binom{n-k}{k}.$$
This means that in a certain sense there are as many k-forms as $n-k$-forms. In fact, there is a natural way to turn k-forms into $n-k$-forms. This is the *Hodge star operator*. Hodge star of α is denoted by $*\alpha$ (or sometimes α^*) and is defined as follows. If $\alpha = \sum_I f_I \, dx_I$, then
$$*\alpha = \sum_I f_I(*dx_I),$$
with
$$*dx_I = \varepsilon_I \, dx_{I^c}.$$
Here, for any increasing multi-index I, I^c denotes the *complementary* increasing multi-index. For instance, if $n = 5$ and $I = (2,5)$, then $I^c = (1,3,4)$. The factor ε_I is a sign,
$$\varepsilon_I = \begin{cases} +1 & \text{if } dx_I \, dx_{I^c} = dx_1 \, dx_2 \cdots dx_n \\ -1 & \text{if } dx_I \, dx_{I^c} = -dx_1 \, dx_2 \cdots dx_n. \end{cases}$$
Put more simply, $*dx_I$ is defined in such a way that it contains exactly those variables that do not occur in dx_I and the product $dx_I(*dx_I)$ is the volume form:
$$dx_I(*dx_I) = dx_1 \, dx_2 \cdots dx_n.$$

4.8. EXAMPLE. On $\mathbf{R}^2$ we have $*dx = dy$ and $*dy = -dx$. On $\mathbf{R}^3$ we have
$$\begin{aligned} *dx &= dy\,dz & \qquad *(dx\,dy) &= dz \\ *dy &= -dx\,dz = dz\,dx & *(dx\,dz) &= -dy \\ *dz &= dx\,dy & *(dy\,dz) &= dx. \end{aligned}$$
(This is the reason that 2-forms on $\mathbf{R}^3$ are sometimes written as $f\,dx\,dy + g\,dz\,dx + h\,dy\,dz$, in contravention of our usual rule to write the variables in increasing order. In higher dimensions it is better to stick to the rule.) On $\mathbf{R}^4$ we have
$$\begin{aligned} *dx_1 &= dx_2\,dx_3\,dx_4 & \qquad *dx_3 &= dx_1\,dx_2\,dx_4 \\ *dx_2 &= -dx_1\,dx_3\,dx_4 & *dx_4 &= -dx_1\,dx_2\,dx_3, \end{aligned}$$
and
$$\begin{aligned} *(dx_1\,dx_2) &= dx_3\,dx_4 & \qquad *(dx_2\,dx_3) &= dx_1\,dx_4 \\ *(dx_1\,dx_3) &= -dx_2\,dx_4 & *(dx_2\,dx_4) &= -dx_1\,dx_3 \\ *(dx_1\,dx_4) &= dx_2\,dx_3 & *(dx_3\,dx_4) &= dx_1\,dx_2. \end{aligned}$$
On $\mathbf{R}^n$ we have $*1 = dx_1\,dx_2 \cdots dx_n$, $*(dx_1\,dx_2 \cdots dx_n) = 1$, and
$$\begin{aligned} *dx_i &= (-1)^{i+1} dx_1\,dx_2 \cdots \widehat{dx_i} \cdots dx_n & \qquad &\text{for } 1 \leq i \leq n, \\ *(dx_i\,dx_j) &= (-1)^{i+j+1} dx_1\,dx_2 \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots dx_n & &\text{for } 1 \leq i < j \leq n. \end{aligned}$$

div, grad, curl

A *vector field* on an open subset U of $\mathbf{R}^n$ is simply a smooth map $F\colon U \to \mathbf{R}^n$. We can write F in components as

$$F(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix},$$

or alternatively as $F = \sum_{i=1}^n F_i \mathbf{e}_i$. We can turn F into a 1-form α by using the F_i as coefficients: $\alpha = \sum_{i=1}^n F_i\,dx_i$. Clearly F is determined by α and vice versa. Thus vector fields and 1-forms are symbiotically associated to one another.

$$\boxed{\text{vector fields} \longleftrightarrow \text{1-forms:} \qquad \sum_{i=1}^n F_i \mathbf{e}_i \longleftrightarrow \sum_{i=1}^n F_i\,dx_i}$$

Likewise a scalar-valued function $f\colon U \to \mathbf{R}$ is associated with an n-form, namely $f\,dx_1\,dx_2 \cdots dx_n$.

$$\boxed{\text{functions} \longleftrightarrow n\text{-forms:} \qquad f \longleftrightarrow f\,dx_1\,dx_2 \cdots dx_n}$$

We can combine this process with exterior differentiation. For each function f the 1-form $df = \sum_{i=1}^n (\partial f/\partial x_i)\,dx_i$ is associated to the vector field

$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathbf{e}_i = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

This vector field is called the *gradient* of f. (Equivalently we can view $\operatorname{grad} f$ as the transpose of the Jacobi matrix of f.)

$$\boxed{\text{grad } f \text{ is the vector field associated with the 1-form } df.}$$

Throwing in the Hodge star gives some interesting combinations. Starting with a vector field $F = \sum_{i=1}^n F_i \mathbf{e}_i$ and letting $\alpha = \sum_{i=1}^n F_i\,dx_i$ we have

$$*\alpha = \sum_{i=1}^n F_i(*dx_i) = \sum_{i=1}^n F_i(-1)^{i+1}\,dx_1\,dx_2 \cdots \widehat{dx_i} \cdots dx_n,$$

so

$$\begin{aligned} d(*\alpha) &= \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(-1)^{i+1}\,dx_i\,dx_1\,dx_2 \cdots \widehat{dx_i} \cdots dx_n \\ &= \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}\,dx_1\,dx_2 \cdots dx_i \cdots dx_n \\ &= \left(\sum_{i=1}^n \frac{\partial F_i}{\partial x_i}\right) dx_1\,dx_2 \cdots dx_n. \end{aligned}$$

The function $\operatorname{div} F = \sum_{i=1}^n \partial F_i/\partial x_i$ is the *divergence* of F. Thus if α is the 1-form associated with F, then

$$\boxed{\operatorname{div} F \text{ is the function associated with the } n\text{-form } d(*\alpha).}$$

On the other hand

$$d\alpha = \sum_{i,j=1}^n \frac{\partial F_i}{\partial x_j} dx_j\, dx_i = \sum_{1 \le i < j \le n} \Bigl(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\Bigr) dx_i\, dx_j.$$

Hence

$$*(d\alpha) = \sum_{1 \le i < j \le n} (-1)^{i+j+1} \Bigl(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\Bigr) dx_1\, dx_2 \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots dx_n.$$

In three dimensions $*(d\alpha)$ is a 1-form and so is associated to a vector field, namely

$$\operatorname{curl} F = \Bigl(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\Bigr)\mathbf{e}_1 - \Bigl(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}\Bigr)\mathbf{e}_2 + \Bigl(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\Bigr)\mathbf{e}_3,$$

the *curl* of F. Thus, for $n = 3$, if α is the 1-form associated with F, then

$$\boxed{\operatorname{curl} F \text{ is the vector field associated with the 1-form } *(d\alpha).}$$

You need not memorize every detail of this discussion. The point is rather to remember that exterior differentiation in combination with the Hodge star unifies and extends to arbitrary dimensions the classical differential operators of vector calculus.

Exercises

4.1. Compute the exterior derivative of the following forms. Recall that a hat indicates that a term has to be omitted.

(i) $e^{xyz}dx$.
(ii) $\sum_{i=1}^n x_i^2\, dx_1 \cdots \widehat{dx_i} \cdots dx_n$.
(iii) $\|\mathbf{x}\|^p \sum_{i=1}^n (-1)^{i+1} x_i\, dx_1 \cdots \widehat{dx_i} \cdots dx_n$, where p is a real constant. For what values of p is this form closed?

4.2. Consider the forms $\alpha = x\,dx - y\,dy$, $\beta = z\,dx\,dy + x\,dy\,dz$ and $\gamma = z\,dy$ on $\mathbf{R}^3$. Calculate

(i) $\alpha\beta$, $\alpha\beta\gamma$;
(ii) $d\alpha$, $d\beta$, $d\gamma$;
(iii) $*\alpha$, $*\beta$, $*\gamma$, $*(\alpha\beta)$.

4.3. Consider the form $\alpha = -x_1\, dx_1 + x_2\, dx_2$ on $\mathbf{R}^2$.

(i) Find $*\alpha$ and $*d(*d\alpha)$.
(ii) Repeat the calculation, regarding α as a form on $\mathbf{R}^3$.
(iii) Again repeat the calculation, now regarding α as a form on $\mathbf{R}^4$.

4.4. Write the coordinates on $\mathbf{R}^{2n}$ as $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$. Let

$$\omega = dx_1\, dy_1 + dx_2\, dy_2 + \cdots + dx_n\, dy_n = \sum_{i=1}^n dx_i\, dy_i.$$

Compute $\omega^n = \omega\omega\cdots\omega$ (n-fold product). First work out the cases $n = 1, 2, 3$.

4.5. Write the coordinates on $\mathbf{R}^{2n+1}$ as $(x_1, y_1, x_2, y_2, \dots, x_n, y_n, z)$. Let

$$\alpha = dz + x_1\,dy_1 + x_2\,dy_2 + \cdots + x_n\,dy_n = dz + \sum_{i=1}^{n} x_i\,dy_i.$$

Compute $\alpha(d\alpha)^n = \alpha(d\alpha d\alpha \cdots d\alpha)$. First work out the cases $n = 1, 2, 3$.

4.6. Let $\alpha = \sum_{i=1}^{n} f_i\,dx_i$ be a *closed* C^∞ 1-form on $\mathbf{R}^n$. Define a function g by

$$g(\mathbf{x}) = \int_0^{x_1} f_1(t, x_2, x_3, \dots, x_n)\,dt + \int_0^{x_2} f_2(0, t, x_3, x_4, \dots, x_n)\,dt$$
$$+ \int_0^{x_3} f_3(0, 0, t, x_4, x_5, \dots, x_n)\,dt + \cdots + \int_0^{x_n} f_n(0, 0, \dots, 0, t)\,dt.$$

Show that $dg = \alpha$. (Apply the fundamental theorem of calculus, differentiate under the integral sign and don't forget to use $d\alpha = 0$.)

4.7. Check that each of the following forms $\alpha \in \Omega^1(\mathbf{R}^3)$ is closed and find a function g such that $dg = \alpha$.

(i) $\alpha = (ye^{xy} - z\sin(xz))\,dx + (xe^{xy} + z^2)\,dy + (-x\sin(xz) + 2yz + 3z^2)\,dz$;
(ii) $\alpha = 2xy^3z^4\,dx + (3x^2y^2z^4 - ze^y\sin(ze^y))\,dy + (4x^2y^3z^3 - e^y\sin(ze^y) + e^z)\,dz$.

4.8. Let $\alpha = \sum_{i=1}^{n} f_i\,dx_i$ be a *closed* 1-form such that all the coefficients f_i are homogeneous functions of the same degree k. Let

$$g(\mathbf{x}) = \frac{1}{k+1}\sum_{i=1}^{n} x_i f_i(\mathbf{x}).$$

Show that $dg = \alpha$. (Use $d\alpha = 0$ and apply the identity proved in Exercise 2.4 to each f_i.)

4.9. Let α and β be closed forms. Prove that $\alpha\beta$ is also closed.

4.10. Let α be closed and β exact. Prove that $\alpha\beta$ is exact.

4.11. Prove that $**\alpha = (-1)^{kn+k}\alpha$ for every k-form α.

4.12. Let $\alpha = \sum_I a_I\,dx_I$ and $\beta = \sum_I b_I\,dx_I$ be *constant* k-forms, i.e. with constant coefficients a_I and b_I. (We also assume, as usual, that the multi-indices I are increasing.) The *inner product* of α and β is the number defined by

$$(\alpha, \beta) = \sum_I a_I b_I.$$

Prove the following assertions.

(i) The dx_I form an orthonormal basis of the space of constant k-forms.
(ii) $(\alpha, \alpha) \geq 0$ for all α and $(\alpha, \alpha) = 0$ if and only if $\alpha = 0$.
(iii) $\alpha(*\beta) = (\alpha, \beta)\,dx_1\,dx_2 \cdots dx_n$.
(iv) $\alpha(*\beta) = \beta(*\alpha)$.
(v) The Hodge star operator is orthogonal, i.e. $(\alpha, \beta) = (*\alpha, *\beta)$.

4.13. The *Laplacian* of a smooth function on an open subset of $\mathbf{R}^n$ is defined by

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Prove the following formulas.

(i) $\Delta f = *(d(*df))$.
(ii) $\Delta(fg) = (\Delta f)g + f\Delta g + 2*(df(*dg))$. (Use Exercise 4.12(iv).)

4.14. The *Minkowski* or *relativistic* inner product on $\mathbf{R}^{n+1}$ is given by

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

A vector $\mathbf{x} \in \mathbf{R}^{n+1}$ is *spacelike* if $(\mathbf{x}, \mathbf{x}) > 0$, *lightlike* if $(\mathbf{x}, \mathbf{x}) = 0$, and *timelike* if $(\mathbf{x}, \mathbf{x}) < 0$.

(i) Give examples of (nonzero) vectors of each type.
(ii) Show that for every $\mathbf{x} \neq 0$ there is a $\mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) \neq 0$.

A Hodge star operator corresponding to this inner product is defined as follows: if $\alpha = \sum_I f_I \, dx_I$, then

$$*\alpha = \sum_I f_I (*dx_I),$$

with

$$*dx_I = \begin{cases} \varepsilon_I \, dx_{I^c} & \text{if } I \text{ contains } n+1 \\ -\varepsilon_I \, dx_{I^c} & \text{if } I \text{ does not contain } n+1. \end{cases}$$

(Here ε_I and I^c are as in the definition of the ordinary Hodge star.)

(iii) Find $*1$, $*dx_i$ for $1 \leq i \leq n+1$, and $*(dx_1 \cdots dx_n)$.
(iv) Compute the "relativistic Laplacian" (usually called the d'Alembertian or wave operator) $*d(*df))$ for any smooth function f on $\mathbf{R}^{n+1}$.
(v) For $n = 3$ (ordinary space-time) find $*(dx_i \, dx_j)$ for $1 \leq i < j \leq 4$.

4.15. One of the greatest advances in theoretical physics of the nineteenth century was Maxwell's formulation of the equations of electrodynamics:

$$\begin{aligned} &\operatorname{curl} E = -\frac{1}{c}\frac{\partial B}{\partial t} && \text{(Faraday's Law)}, \\ &\operatorname{curl} H = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial D}{\partial t} && \text{(Ampère's Law)}, \\ &\operatorname{div} D = 4\pi\rho && \text{(charge is source of electric field)}, \\ &\operatorname{div} B = 0 && \text{(no magnetic monopoles)}. \end{aligned}$$

Here c is the speed of light, E is the electric field, H is the magnetic field, J is the density of electric current, ρ is the density of electric charge, B is the magnetic induction and D is the dielectric displacement. E, H, J, B and D are vector fields and ρ is a function on $\mathbf{R}^3$ and all depend on time t. The Maxwell equations look even prettier in differential form notation, as we shall now see. In space-time $\mathbf{R}^4$ with coordinates (x_1, x_2, x_3, x_4), where $x_4 = ct$, introduce forms

$$\begin{aligned} \alpha &= (E_1 \, dx_1 + E_2 \, dx_2 + E_3 \, dx_3) \, dx_4 + B_1 \, dx_2 \, dx_3 + B_2 \, dx_3 \, dx_1 + B_3 \, dx_1 \, dx_2, \\ \beta &= -(H_1 \, dx_1 + H_2 \, dx_2 + H_3 \, dx_3) \, dx_4 + D_1 \, dx_2 \, dx_3 + D_2 \, dx_3 \, dx_1 + D_3 \, dx_1 \, dx_2, \\ \gamma &= \frac{1}{c}(J_1 \, dx_2 \, dx_3 + J_2 \, dx_3 \, dx_1 + J_3 \, dx_1 \, dx_2) \, dx_4 - \rho \, dx_1 \, dx_2 \, dx_3. \end{aligned}$$

(i) Show that Maxwell's equations are equivalent to

$$d\alpha = 0,$$
$$d\beta + 4\pi\gamma = 0.$$

(ii) Conclude that γ is closed and that $\operatorname{div} J + \partial\rho/\partial t = 0$.
(iii) *Free space* is a vacuum without charges or currents. In vacuum one has $E = D$ and $H = B$. Show that Maxwell's equations in free space reduce to

$$\begin{aligned} &\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}, && \operatorname{div} E = 0, \\ &\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t}, && \operatorname{div} H = 0, \end{aligned}$$

or equivalently: $d\alpha = d\beta = 0$.

(iv) Show that in free space $\beta = *\alpha$, the relativistic Hodge star of α. (A form ω is called *co-closed* if ω is closed. Thus the free Maxwell equations are equivalent to α being closed and co-closed.)