CHAPTER 6

Integration of 1-forms

Like functions, forms can be integrated as well as differentiated. Differentiation and integration are related via multivariable version of the fundamental theorem of calculus, known as Stokes' theorem. In this chapter we investigate the case of 1-forms.

1. Definition and elementary properties of the integral

Let U be an open subset of $\mathbf{R}^n$. A *parametrized curve* in U is a smooth mapping $c\colon I \to U$ from an interval I into U. We want to integrate over I and to avoid problems with improper integrals we assume I to be closed and bounded, $I = [a, b]$. (Strictly speaking we have not defined what we mean by a smooth map $c\colon [a, b] \to U$. The easiest definition is that c should be the restriction of a smooth map $\tilde{c}\colon (a - \varepsilon, b + \varepsilon) \to U$ defined on a slightly larger *open* interval.) Let α be a 1-form on U. The pullback $c^*\alpha$ is a 1-form on $[a, b]$, and can therefore be written as $c^*\alpha = g\,dt$ (where t is the coordinate on $\mathbf{R}$). The *integral* of α over c is now defined by

$$\int_c \alpha = \int_{[a,b]} c^*\alpha = \int_a^b g(t)\,dt.$$

More explicitly, writing α in components, $\alpha = \sum_{i=1}^n f_i\,dx_i$, we have

$$c^*\alpha = \sum_{i=1}^n (c^*f_i)\,dc_i = \sum_{i=1}^n (c^*f_i)\frac{dc_i}{dt}\,dt,$$

so

$$\int_c \alpha = \sum_{i=1}^n \int_a^b f_i(c(t))\frac{dc_i}{dt}(t)\,dt.$$

6.1. Example. Let U be the punctured plane $\mathbf{R}^2 - \{\mathbf{0}\}$. Let $c\colon [0, 2\pi] \to U$ be the usual parametrization of the circle, $c(t) = (\cos t, \sin t)$, and

$$\alpha = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

Then $c^*\alpha = dt$ (see Example 5.7), so $\int_c \alpha = \int_0^{2\pi} dt = 2\pi$.

A curve $c\colon [a, b] \to U$ can be *reparametrized* by substituting a new variable, $t = p(s)$. We shall require the function p to be a diffeomorphism from another interval $[\bar{a}, \bar{b}]$ onto $[a, b]$. (Recall that this means that p is smooth and has a smooth inverse $p^{-1}\colon [a, b] \to [\bar{a}, \bar{b}]$.) Such a p is called a *reparametrization*. The curve

$$c \circ p\colon [\bar{a}, \bar{b}] \to U$$

has the same image as the original curve c, but it is traversed at a different rate. Being a diffeomorphism, p has to be either increasing or decreasing. In fact, since $p(s) \neq 0$ for all $s \in [\bar{a}, \bar{b}]$, so either $p'(s) > 0$ for all s or $p'(s) < 0$ for all s. If p is increasing, we say that it *preserves* the orientation of the curve (or that the curves c and $c \circ p$ have the *same orientation*); if p is decreasing, we say that it *reverses* the orientation (or that c and $c \circ p$ have *opposite orientations*). In the orientation-reversing case, $c \circ p$ traverses the curve in the opposite direction to c.

6.2. EXAMPLE. The curve $c\colon [0, 2\pi] \to \mathbf{R}^2$ defined by $c(t) = (\cos t, \sin t)$ represents the unit circle in the plane, traversed at a constant rate (angular velocity) of 1 radian per second. Let $p(s) = 2s$. Then p maps $[0, \pi]$ to $[0, 2\pi]$ and $c \circ p$, regarded as a map $[0, \pi] \to \mathbf{R}^2$, represents the same circle, but traversed at 2 radians per second. (It is important to restrict the domain of p to the interval $[0, \pi]$. If we allowed s to range over $[0, 2\pi]$, then $(\cos 2s, \sin 2s)$ would traverse the circle twice. This is not considered a reparametrization of the original curve c.) Now let $p(s) = -s$. Then $c \circ p\colon [0, 2\pi] \to \mathbf{R}^2$ traverses the unit circle in the clockwise direction. This reparametrization reverses the orientation; the angular velocity is now -1 radian per second. Finally let $p(s) = 2\pi s^2$. Then p maps $[0, 1]$ to $[0, 2\pi]$ and $c \circ p\colon [0, 1] \to \mathbf{R}^2$ runs once counterclockwise through the unit circle, but at a variable rate. What is the angular velocity as a function of s?

It turns out that the integral of a form along a curve is almost completely independent of the parametrization.

6.3. THEOREM. *Let α be a 1-form on U and $c\colon [a, b] \to U$ a curve in U. Let $p\colon [\bar{a}, \bar{b}] \to [a, b]$ be a reparametrization. Then*

$$\int_{c \circ p} \alpha = \begin{cases} \int_c \alpha & \textit{if } p \textit{ preserves the orientation} \\ -\int_c \alpha & \textit{if } p \textit{ reverses the orientation.} \end{cases}$$

PROOF. It follows from the definition of the integral and from the naturality of pullbacks (Proposition 5.10(iii)) that

$$\int_{c \circ p} \alpha = \int_{[\bar{a}, \bar{b}]} (c \circ p)^* \alpha = \int_{[\bar{a}, \bar{b}]} p^*(c^* \alpha).$$

Now let us write $c^*\alpha = g\,dt$ and $t = p(s)$. Then $p^*(c^*\alpha) = p^*(g\,dt) = (p^*g)\,dp = (p^*g)(dp/ds)\,ds$, so

$$\int_{c \circ p} \alpha = \int_{[\bar{a}, \bar{b}]} (p^*g) \frac{dp}{ds} ds = \int_{\bar{a}}^{\bar{b}} g(p(s))p'(s)\,ds.$$

On the other hand, $\int_c \alpha = \int_a^b g(t)\,dt$, so by the substitution formula, Theorem 2.7, we have $\int_{c \circ p} \alpha = \pm \int_c \alpha$, where the $+$ occurs if $p' > 0$ and the $-$ if $p' < 0$. QED

Interpretation of the integral. Integrals of 1-forms play an important role in physics and engineering. A curve $c\colon [a, b] \to U$ models a particle travelling through the region U. To a 1-form $\alpha = \sum_{i=1}^n f_i\,dx_i$ corresponds a vector field $F = \sum_{i=1}^n f_i \mathbf{e}_i$, which can be thought of as a force field acting on the particle. The

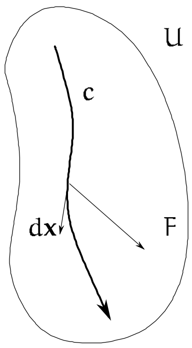
dx's are often written formally as the components of a vector,

$$d\mathbf{x} = \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix},$$

which can be interpreted as an infinitesimal vector tangent to the curve. With these conventions we can view α as an inner product,

$$\alpha = \sum_{i=1}^{n} f_i\,dx_i = F \cdot d\mathbf{x}.$$

Thus α represents the work done by the force field along an infinitesimal vector $d\mathbf{x}$. The integral $\int_c \alpha = \int_c F \cdot dr$ is then the total work done by the force on the particle during its trip along c.



Theorem 6.3 can be translated into this language as follows: the work done by the force does not depend on the rate at which the particle speeds along its path, but only on the path itself and on the direction of travel.

The field F is *conservative* if it can be written as the gradient of a function, $F = \operatorname{grad} g$. The function $-g$ is called a *potential* for the field and is interpreted as the potential energy of the particle. In terms of forms this means that $\alpha = dg$, i.e. α is exact.

2. Integration of exact 1-forms

Integrating an exact 1-form $\alpha = dg$ is easy once the function g is known.

6.4. THEOREM (fundamental theorem of calculus). *Let $\alpha = dg$ be an exact 1-form on an open subset U of $\mathbf{R}^n$. Let $c\colon [a,b] \to U$ be a parametrized curve. Then*

$$\boxed{\int_c \alpha = g(c(b)) - g(c(a)).}$$

PROOF. By Theorem 5.11 we have $c^*\alpha = c^*dg = dc^*g$. Writing $h(t) = c^*g(t) = g(c(t))$ we have $c^*\alpha = dh$, so

$$\int_c \alpha = \int_{[a,b]} c^*\alpha = \int_a^b dh = h(b) - h(a),$$

where we used the (ordinary) fundamental theorem of calculus. Hence $\int_c \alpha = g(c(b)) - g(c(a))$. QED

The physical interpretation of this result is that when a particle moves in a conservative force field, its potential energy decreases by the amount of work done by the field. This clarifies what it means for a field to be conservative: it means that the work done is entirely converted into mechanical energy and that none is dissipated by friction into heat, radiation, etc. Thus the FTC "explains" the law of conservation of energy.

This theorem yields a necessary and sufficient criterion for a 1-form on U to be exact. A curve $c\colon [a,b] \to U$ is called *closed* if $c(a) = c(b)$.

6.5. THEOREM. *Let α be a 1-form on an open subset U of $\mathbf{R}^n$. Then the following statements are equivalent.*

(i) *α is exact;*
(ii) *$\int_c \alpha = 0$ for all closed curves c;*
(iii) *$\int_c \alpha$ depends only on the endpoints of c for every curve c in U.*

PROOF. (i) $\implies$ (ii): if $\alpha = dg$ and c is closed, then $\int_c \alpha = g(c(b)) - g(c(a)) = 0$ by the fundamental theorem of calculus.

(ii) $\implies$ (iii): assume $\int_c \alpha = 0$ for all closed curves c. Let

$$c_1\colon [a_1, b_1] \to U \qquad \text{and} \qquad c_2\colon [a_2, b_2] \to U$$

be two curves with the same endpoints, i.e. $c_1(a_1) = c_2(a_2)$ and $c_1(b_1) = c_2(b_2)$. We need to show that $\int_{c_1} \alpha = \int_{c_2} \alpha$. After reparametrizing c_1 and c_2 we may assume that $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$. Define a new curve c by

$$c(t) = \begin{cases} c_1(t) & \text{for } 0 \le t \le 1 \\ c_2(2-t) & \text{for } 1 \le t \le 2. \end{cases}$$

(First traverse c_1, then traverse c_2 backwards.) Then c is closed, so $\int_c \alpha = 0$. But Theorem 6.3 implies $\int_c \alpha = \int_{c_1} \alpha - \int_{c_2} \alpha$, so $\int_{c_1} \alpha = \int_{c_2} \alpha$.

(iii) $\implies$ (i): assume that, for all c, $\int_c \alpha$ depends only on the endpoints of c. We must define a function g such that $\alpha = dg$. Fix a point $\mathbf{x}_0$ in U. For each point $\mathbf{x}$ in U choose a curve $c_{\mathbf{x}}\colon [0,1] \to U$ which joins $\mathbf{x}_0$ to $\mathbf{x}$. Define

$$g(\mathbf{x}) = \int_{c_{\mathbf{x}}} \alpha.$$

We assert that dg is well-defined and equal to α. Write $\alpha = \sum_{i=1}^n f_i\, dx_i$. We must show that $\partial g/\partial x_i = f_i$. From the definition of partial differentiation,

$$\frac{\partial g}{\partial x_i}(\mathbf{x}) = \lim_{h\to 0} \frac{g(\mathbf{x}+h\mathbf{e}_i) - g(\mathbf{x})}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_{c_{\mathbf{x}+h\mathbf{e}_i}} \alpha - \int_{c_{\mathbf{x}}} \alpha\right).$$

Now consider a curve $\tilde{c}$ composed of two pieces: for $0 \le t \le 1$ travel from $\mathbf{x}_0$ to $\mathbf{x}$ along the curve $c_{\mathbf{x}}$ and then for $1 \le t \le 2$ travel from $\mathbf{x}$ to $\mathbf{x} + h\mathbf{e}_i$ along the straight line given by $l(t) = \mathbf{x} + (t-1)h\mathbf{e}_i$. Then $\tilde{c}$ has the same endpoints as

$c_{\mathbf{x}+h\mathbf{e}_i}$. Therefore $\int_{c_{\mathbf{x}+h\mathbf{e}_i}} \alpha = \int_{\tilde{c}} \alpha$, and hence

$$\begin{aligned}
\frac{\partial g}{\partial x_i}(\mathbf{x}) &= \lim_{h\to 0} \frac{1}{h}\left(\int_{\tilde{c}} \alpha - \int_{c_{\mathbf{x}}} \alpha\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_{c_{\mathbf{x}}} \alpha + \int_{l} \alpha - \int_{c_{\mathbf{x}}} \alpha\right) \\
&= \lim_{h\to 0} \frac{1}{h}\int_{l} \alpha \\
&= \lim_{h\to 0} \frac{1}{h}\int_{[1,2]} l^*\alpha \\
&= \lim_{h\to 0} \frac{1}{h}\int_1^2 \sum_{j=1}^n f_j(\mathbf{x} + (t-1)h\mathbf{e}_i)h\,dt \qquad (\text{use } l_j(t) = x_j + \delta_{ij}(t-1)h) \\
&= \int_0^1 \lim_{h\to 0} f_i(\mathbf{x} + th\mathbf{e}_i)dt \\
&= f_i(\mathbf{x}).
\end{aligned}$$

Here δ_{ij} is the *Kronecker delta* defined by $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. This proves that g is smooth and that $dg = \alpha$. QED

This theorem, and its proof, can be used in many different ways. For example, it tells us that once we know a 1-form α to be exact we can find an "antiderivative" $g(\mathbf{x})$ by integrating α along an arbitrary path running from a fixed point $\mathbf{x}_0$ to $\mathbf{x}$. On the other hand, the theorem also enables us to detect closed 1-forms that are not exact.

6.6. EXAMPLE. The closed form $\alpha \in \Omega^1(\mathbf{R}^2 - \{\mathbf{0}\})$ of Example 6.1 is not exact. Indeed, its integral around the circle is $2\pi \neq 0$. Mark the contrast with closed 1-forms on $\mathbf{R}^n$, which are always exact! (See Exercise 4.6.) This phenomenon underlines the importance of being careful about the domain of definition of a form.

The global angle function and the winding number

It is worthwhile having a closer look at the form of Examples 6.1 and 6.6 as it carries interesting information of a "topological" nature. Let U be the punctured plane $\mathbf{R}^2 - \{0\}$ and put

$$\alpha = \frac{-y\,dx + x\,dy}{x^2+y^2}, \qquad \xi = \frac{x}{\sqrt{x^2+y^2}}, \qquad \eta = \frac{y}{\sqrt{x^2+y^2}}.$$

Then α is a closed 1-form, and ξ and η are smooth functions on U. In fact ξ and η are just the components of $\mathbf{x}/\|\mathbf{x}\|$, the unit vector pointing in the direction of $\mathbf{x}$. These functions satisfy

$$\alpha = \xi d\eta - \eta d\xi. \tag{6.1}$$

(Proof as an exercise.) Now let $\theta \colon U \to [0, 2\pi)$ be the angle between a point and the positive x-axis, chosen to lie in the interval $[0, 2\pi)$. Then $\xi = \cos\theta$ and $\eta = \sin\theta$, so by (6.1)

$$\alpha = \cos\theta\, d\sin\theta - \sin\theta\, d\cos\theta = \cos^2\theta\, d\theta + (-\sin\theta)^2\, d\theta = d\theta.$$

This equation is not valid on all of U (it cannot be because α is not exact), but only where θ is differentiable, i.e. on the complement of the positive x-axis. Hence the nonexactness of α is closely related to the impossibility of defining a global differentiable angle function on U. (The precise meaning of this assertion will become clear in Exercise 6.5.)

However, along a curve $c\colon [a,b] \to U$ we can define a continuous angle function, and the fact that $\alpha = d\theta$ almost everywhere suggests how: by integrating α along c. For simplicity assume that $a = 0$ and $b = 1$. Start by fixing any ϑ_0 such that $\cos\vartheta_0 = \xi(c(0))$ and $\sin\vartheta_0 = \eta(c(0))$ and then define

$$\vartheta(t) = \vartheta_0 + \int_{[0,t]} c^*\alpha.$$

The following result says that $\vartheta(t)$ measures the angle between $c(t)$ and the positive x-axis (up to an integer multiple of 2π) and that the function $\vartheta\colon [0,1] \to \mathbf{R}$ is smooth. In this sense ϑ is a "differentiable choice of angle" along the curve c.

6.7. THEOREM. *The function ϑ is smooth and satisfies*

$$\vartheta(0) = \vartheta_0, \quad \cos\vartheta(t) = \xi(c(t)) \quad \textit{and} \quad \sin\vartheta(t) = \eta(c(t)).$$

PROOF. The equality $\vartheta(0) = \vartheta_0$ is clear. Write $f(t) = \xi(c(t))$ and $g(t) = \eta(c(t))$. That is, $f(t)$ and $g(t)$ are the components of the unit vector $c(t)/\|c(t)\|$. Then $f = c^*\xi$ and $g = c^*\eta$, so it follows from (6.1) that $c^*\alpha = f\,dg - g\,df$. Therefore

$$\vartheta(t) = \vartheta_0 + \int_0^t \bigl(f(s)g'(s) - g(s)f'(s)\bigr)\,ds.$$

By the fundamental theorem of calculus ϑ is differentiable and

$$\vartheta' = fg' - gf'. \tag{6.2}$$

Since the right-hand side is smooth, ϑ is smooth as well. To establish that $\cos\vartheta(t) = f(t)$ and $\sin\vartheta(t) = g(t)$ for all t it is enough to show that the difference vector

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} - \begin{pmatrix} \cos\vartheta(t) \\ \sin\vartheta(t) \end{pmatrix}$$

has length 0. Its length is equal to

$$\begin{aligned}(f-\cos\vartheta)^2 + (g-\sin\vartheta)^2 &= f^2 + g^2 - 2(f\cos\vartheta + g\sin\vartheta) + \cos^2\vartheta + \sin^2\vartheta \\ &= 2 - 2(f\cos\vartheta + g\sin\vartheta).\end{aligned}$$

Hence we need to show that the function $u = f\cos\vartheta + g\sin\vartheta$ is a constant equal to 1. For $t = 0$ we have

$$u(0) = f(0)\cos\vartheta_0 + g(0)\sin\vartheta_0 = \cos^2\vartheta_0 + \sin^2\vartheta_0 = 1.$$

Furthermore, the derivative of u is

$$\begin{aligned} u' &= f'\cos\vartheta - f\vartheta'\sin\vartheta + g'\sin\vartheta + g\vartheta'\cos\vartheta \\ &= (f' - g^2f' + fgg')\cos\vartheta + (g' - f^2g' + fgf')\sin\vartheta && \text{by (6.2)} \\ &= (f^2f' + fgg')\cos\vartheta + (g^2g' + fgf')\sin\vartheta && \text{since } f^2 + g^2 = 1 \\ &= f(ff' + gg')\cos\vartheta + g(gg' + ff')\sin\vartheta. \end{aligned}$$

Now $f^2 + g^2 = 1$ implies $ff' + gg' = 0$, so $u'(t) = 0$ for all t. Hence u is a constant function, so $u(t) = 1$ for all t. QED

It is useful to think of the vector $(f(t), g(t)) = c(t)/\|c(t)\|$ as a "dial", which is mounted on a circular "gauge" or "meter" of radius 1 centred at the origin. The dial points in the same direction as the vector $c(t)$, so the meter monitors the angle between the positive x-axis and a particle travelling along the path $c(t)$. As t increases from 0 to 1, the dial starts at the angle $\vartheta(0)$, it moves around the meter, and ends up at the final angle $\vartheta(1)$. The difference $\vartheta(1) - \vartheta(0)$ measures the total angle swept out by the dial.

6.8. COROLLARY. *If $c\colon [0, 1] \to U$ is closed, then $\vartheta(1) - \vartheta(0) = 2\pi k$, where k is an integer.*

PROOF. By Theorem 6.7, $c(0) = c(1)$ implies

$$\big(\cos\vartheta(0), \sin\vartheta(0)\big) = \big(\xi(c(0)), \eta(c(0))\big) = \big(\xi(c(1)), \eta(c(1))\big) = \big(\cos\vartheta(1), \sin\vartheta(1)\big).$$

Hence $\vartheta(0)$ and $\vartheta(0)$ differ by an integer multiple of 2π. QED

The integer $k = (2\pi)^{-1} \int_c \alpha$ is called the *winding number* of the closed curve c about the origin. It measures how many times the curve loops around the origin.

6.9. EXAMPLE. The winding number of the circle $c(t) = (\cos t, \sin t)$ $(0 \leq t \leq 2\pi)$ is equal to 1.

Exercises

6.1. Restate Theorem 6.5 in terms of force fields, potentials and energy. Explain why the result is plausible on physical grounds.

6.2. Consider the 1-form $\alpha = \|\mathbf{x}\|^a \sum_{i=1}^n x_i\,dx_i$ on $\mathbf{R}^n - \{\mathbf{0}\}$, where a is a real constant. For every $\mathbf{x} \neq 0$ let $c_{\mathbf{x}}$ be the line segment starting at $\mathbf{x}$ and ending at the origin.

(i) Show that α is closed for any value of a.
(ii) Determine for which values of a the function $g(\mathbf{x}) = \int_{c_{\mathbf{x}}} \alpha$ is well-defined and compute it.
(iii) For the values of a you found in (ii) check that $dg = \alpha$.

6.3. Let $\alpha \in \Omega^1(\mathbf{R}^n - \{\mathbf{0}\})$ be the 1-form of Exercise 6.2. Now let $c_{\mathbf{x}}$ be the halfline starting at $\mathbf{x}$ which points radially outward to infinity.

(i) Determine for which values of a the function $g(\mathbf{x}) = \int_{c_{\mathbf{x}}} \alpha$ is well-defined and compute it.
(ii) For the values of a you found in (i) check that $dg = \alpha$.
(iii) Show how to recover from this computation the potential energy for Newton's gravitational force. (See Exercise 2.3.)

6.4. Let $\alpha \in \Omega^1(\mathbf{R}^n - \{\mathbf{0}\})$ be as in Exercise 6.2. There is one value of a which is not covered by Exercises 6.2 and 6.3. For this value of a find $g \in C^\infty(U)$ such that $dg = \alpha$.

6.5. (i) Verify equation (6.1).
(ii) Let $U = \mathbf{R}^2 - \{\mathbf{0}\}$. Prove that there does not exist a smooth function $\theta\colon U \to \mathbf{R}$ satisfying $\cos\theta(x, y) = x/(x^2 + y^2)$ and $\sin\theta(x, y) = y/(x^2 + y^2)$ for all $(x, y) \in U$. (Argue by contradiction, by letting $\alpha = (-y\,dx + x\,dy)/(x^2 + y^2)$ and showing that $\alpha = d\theta$ if θ was such a function.)

6.6. Calculate directly from the definition the winding number about the origin of the curve $c\colon [0, 2\pi] \to \mathbf{R}^2$ given by $c(t) = (\cos kt, \sin kt)$.

6.7. Let $c\colon [0, 1] \to \mathbf{R}^2 - \{\mathbf{0}\}$ be a closed curve with winding number k. Determine the winding numbers of the following curves $\tilde{c}\colon [0, 1] \to \mathbf{R}^2 - \{\mathbf{0}\}$ by using the formula, and then explain the answer by appealing to geometric intuition.

(i) $\tilde{c}(t) = c(1-t)$;
(ii) $\tilde{c}(t) = \rho(t)c(t)$, where $\rho\colon [0,1] \to (0,\infty)$ is a function satisfying $\rho(0) = \rho(1)$;
(iii) $\tilde{c}(t) = \|c(t)\|^{-1}c(t)$;
(iv) $\tilde{c}(t) = \phi(c(t))$, where $\phi(x,y) = (y,x)$;
(v) $\tilde{c}(t) = \phi(c(t))$, where $\phi(x,y) = \dfrac{1}{x^2+y^2}(x,-y)$.

6.8. For each of the following closed curves $\mathbf{c}\colon [0,2\pi] \to \mathbf{R}^2 - \{\mathbf{0}\}$ set up the integral defining the winding number about the origin. Evaluate the integral if you can (but don't give up too soon). If not, sketch the curve and obtain the answer geometrically.

(i) $\mathbf{c}(t) = (a\cos t, b\sin t)$, where $a > 0$ and $b > 0$;
(ii) $\mathbf{c}(t) = (\cos t - 2, \sin t)$;
(iii) $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$;
(iv) $\mathbf{c}(t) = \big((a\cos t + b)\cos t + (b-a)/2, (a\cos t + b)\sin t\big)$, where $0 < b < a$.

6.9. Let $b > 0$ and $a \neq 0$ be constants with $|a| \neq b$. Define a planar curve $\mathbf{c}\colon [0,2\pi] \to \mathbf{R}^2 - \{\mathbf{0}\}$ by

$$\mathbf{c}(t) = \left((a+b)\cos t + a\cos\frac{a+b}{a}t, (a+b)\sin t + a\sin\frac{a+b}{a}t\right).$$

(i) Sketch the curve $\mathbf{c}$ for $a = \pm b/3$.
(ii) For what values of a and b is the curve closed?
(iii) Assume $\mathbf{c}$ is closed. Set up the integral defining the winding number of $\mathbf{c}$ around the origin and evaluate it. If you get stuck, find the answer geometrically.

6.10. Let U be an open subset of $\mathbf{R}^2$ and let $\phi = (\phi_1, \phi_2)\colon U \to \mathbf{R}^2$ be a C^1 map. The differential form

$$\beta = \frac{\phi_1\, d\phi_2 - \phi_2\, d\phi_1}{\phi_1^2 + \phi_2^2}$$

is well-defined at all points $\mathbf{x}$ of U where $\phi(\mathbf{x}) \neq 0$. Let D be a disc contained in U and let $\mathbf{c}$ be the circle bounding D. Assume that $\phi(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{c}$. The *index* of ϕ relative to D is

$$\operatorname{index}(\phi, D) = \frac{1}{2\pi}\int_{\mathbf{c}} \beta,$$

where $\mathbf{c}$ is traversed once in the counterclockwise direction. Prove the following assertions.

(i) $\beta = \phi^*\alpha$, where $\alpha = (-y\,dx + x\,dy)/(x^2+y^2)$;
(ii) β is closed;
(iii) $\operatorname{index}(\phi, D)$ is the winding number of the curve $\phi \circ \mathbf{c}$ about the origin;
(iv) $\operatorname{index}(\phi, D)$ is an integer.

6.11. Let f be a function on $[0,1]$ satisfying $f(0) = f(1)$ and let $\alpha = f\,dt$, where t is the coordinate on $[0,1]$. Show that there is a unique number λ such that $\alpha - \lambda\,dt = dg$ for some function g satisfying $g(0) = g(1)$. (To find λ, integrate the equation $\alpha - \lambda\,dt = dg$ over $[0,1]$.)