CHAPTER 7

Integration and Stokes' theorem

1. Integration of forms over chains

In this chapter we generalize the theory of Chapter 6 to higher dimensions. Let U be an open subset of $\mathbf{R}^n$ and let α be a k-form on U. We can integrate α over the k-dimensional analogue of a parametrized path in the following manner. The analogue of an interval is a *rectangular block* in $\mathbf{R}^k$, which is a set of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k] = \{\, \mathbf{t} \in \mathbf{R}^k : a_i \leq t_i \leq b_i \quad \text{for } 1 \leq i \leq k \,\},$$

where $a_i < b_i$. Let $c\colon R \to U$ be a smooth map. Then $c^*\alpha$ is a k-form on R and therefore looks like $g(\mathbf{t})dt_1\,dt_2 \cdots dt_k$ for some function $g\colon R \to \mathbf{R}$. The *integral* of α over c is defined as

$$\int_c \alpha = \int_R c^*\alpha = \int_{a_k}^{b_k} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} g(\mathbf{t})dt_1\,dt_2 \cdots dt_k.$$

For $k = 1$ this reproduces the definition given in Chapter 6.

The case $k = 0$ is also worth examining. A zero-dimensional "block" R in $\mathbf{R}^0 = \{0\}$ is just the point 0. We can then think of a map $c\colon R \to U$ as a choice of a point $\mathbf{x} = c(0)$ in U. The integral of a 0-form (function) f over c is by definition the value of f at $\mathbf{x}$,

$$\int_c f = f(\mathbf{x}).$$

As in the one-dimensional case, integrals of k-forms are almost wholly unaffected by a change of variables. A *reparametrization* is a diffeomorphism $p\colon \bar{R} \to R$, where

$$\bar{R} = [\bar{a}_1, \bar{b}_1] \times [\bar{a}_2, \bar{b}_2] \times \cdots \times [\bar{a}_k, \bar{b}_k]$$

is a second rectangular block. Then the Jacobi matrix $dp(\mathbf{s})$ is invertible for all $\mathbf{s} \in \bar{R}$, so either $\det dp(\mathbf{s}) > 0$ for all $\mathbf{s}$ or $\det dp(\mathbf{s}) < 0$ for all $\mathbf{s}$. In these cases we say that the reparametrizion *preserves*, respectively *reverses* the orientation of c.

7.1. THEOREM. *Let α be a k-form on U and let $c\colon R \to U$ a smooth map. Let $p\colon \bar{R} \to R$ be a reparametrization. Then*

$$\int_{c \circ p} \alpha = \begin{cases} \int_c \alpha & \textit{if } p \textit{ preserves the orientation} \\ -\int_c \alpha & \textit{if } p \textit{ reverses the orientation.} \end{cases}$$

PROOF. Almost verbatim the same proof as for $k = 1$ (Theorem 6.3). It follows from the definition of the integral and from the contravariance of pullbacks that

$$\int_{c \circ p} \alpha = \int_{\bar{R}} (c \circ p)^*\alpha = \int_{\bar{R}} p^*(c^*\alpha).$$

Now let us write $c^*\alpha = g\,dt_1\,dt_2\cdots dt_k$ and $\mathbf{t} = p(\mathbf{s})$. Then

$$p^*(c^*\alpha) = p^*(g\,dt_1\,dt_2\cdots dt_k) = p^*g\,\det dp\,ds_1\,ds_2\cdots ds_k$$

by Theorem 5.9, so

$$\int_{c\circ p} \alpha = \int_{\bar{R}} g(p(\mathbf{s}))\,\det dp(\mathbf{s})\,ds_1\,ds_2\cdots ds_k.$$

On the other hand, $\int_c \alpha = \int_R g(t)dt_1\,dt_2\cdots dt_k$, so by the substitution formula, Theorem 2.7, we have $\int_{c\circ p} \alpha = \pm\int_c \alpha$, where the $+$ occurs if $\det dp > 0$ and the $-$ if $\det dp < 0$. QED

7.2. EXAMPLE. The *unit interval* is the interval $[0,1]$ in the real line. The *unit cube* in $\mathbf{R}^k$ is the rectangular block

$$[0,1]^k = \{\mathbf{t} \in \mathbf{R}^k : t_i \in [0,1] \quad \text{for } 1 \le i \le k\}.$$

Let R be any other block, given by $a_i \le t_i \le b_i$. Define $p\colon [0,1]^k \to R$ by $p(\mathbf{s}) = A\mathbf{s} + \mathbf{a}$, where

$$A = \begin{pmatrix} b_1 - a_1 & 0 & \dots & 0 \\ 0 & b_2 - a_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & b_k - a_k \end{pmatrix} \qquad \text{and} \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}.$$

("Squeeze the unit cube until it has the same dimensions as R and then move it to the position of R.") Then p is one-to-one and onto and $dp(\mathbf{s}) = A$, so $\det dp(\mathbf{s}) = \det A = \operatorname{vol} R > 0$ for all $\mathbf{s}$, so p is an orientation-preserving reparametrization. Hence $\int_{c\circ p} \alpha = \int_c \alpha$ for any k-form α on$U$.

We see from this example that an integral over any rectangular block can be written as an integral over the unit cube. For this reason, from now on we shall usually take R to be the unit cube. A smooth map $c\colon [0,1]^k \to U$ is called a k-*cube* in U (or sometimes a *singular* k-cube, the word singular meaning that the map c is not assumed to be one-to-one, so that the image can have self-intersections.)

It is often necessary to integrate over regions that are made up of several pieces. A k-*chain* in U is a formal linear combination of k-cubes,

$$c = a_1c_1 + a_2c_2 + \cdots + a_pc_p,$$

where $a_1, a_2, \dots, a_p$ are real coefficients and $c_1, c_2, \dots, c_p$ are k-cubes. For any k-form α we then define

$$\int_c \alpha = \sum_{i=1}^{p} \int_{c_i} \alpha.$$

(In the language of linear algebra, the k-chains form a vector space with a basis consisting of the k-cubes, and operation of integration, which is a priori only defined on cubes, is extended to chains in such a way as to be linear.)

Recall that a 0-cube is nothing but a single point $\mathbf{x}$ in U. Thus a 0-chain is a formal linear combination of points, $c = \sum_{i=1}^{p} a_i\mathbf{x}_i$. A good way to think of this is as a collection of p point charges, with an electric charge a_i placed at the point $\mathbf{x}_i$. The integral of a function f over the 0-chain is

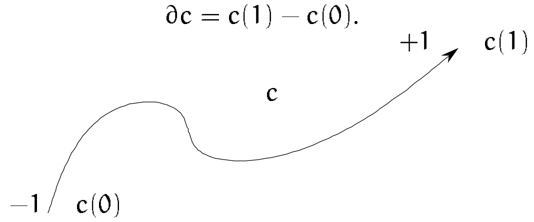
$$\int_c f = \sum_{i=1}^{p} a_i \int_{\mathbf{x}_i} f = \sum_{i=1}^{p} a_i f(\mathbf{x}_i).$$

Likewise, a 1-chain $\sum_{i=1}^{p} c_i$ can be pictured as a distribution of charges, with an electric charge a_i spread along the curve c_i.

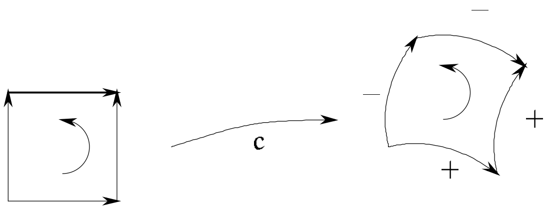
2. The boundary of a chain

Consider a curve ("1-cube") $c\colon [0,1] \to U$. Its *boundary* is by definition the 0-chain defined by

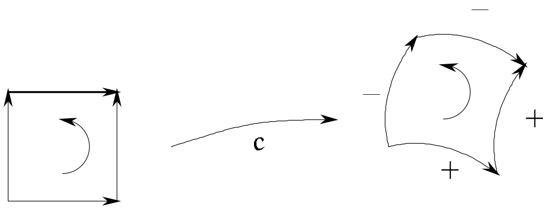
$$\partial c = c(1) - c(0).$$



The boundary of a 2-cube $c\colon [0,1]^2 \to U$ consists of four pieces corresponding to the edges of the unit square: $c_1(t) = c(t,0)$, $c_2(t) = c(1,t)$, $c_3(t) = c(t,1)$ and $c_4(t) = c(0,t)$. The picture below suggests that we should define $\partial c = c_1 + c_2 - c_3 - c_4$.



(Alternatively we could define $\partial c = c_1 + c_2 + \bar{c}_3 + \bar{c}_4$, with $\bar{c}_3(t) = c(1-t,1)$ and $\bar{c}_4(t) = c(0,1-t)$, which corresponds to the following picture:



This would work equally well, but is technically less convenient.)

A k-cube $c\colon [0,1]^k \to X$ has $2k$ faces of dimension $k-1$, which are described as follows. Let $\mathbf{t} = (t_1, t_2, \dots, t_{k-1}) \in [0,1]^{k-1}$ and for $i = 1, 2, \dots, k$ put

$$c_{i,0}(t) = c(t_1, t_2, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1}),$$
$$c_{i,1}(t) = c(t_1, t_2, \dots, t_{i-1}, 1, t_i, \dots, t_{k-1}).$$

("Insert 0, resp. 1 in the ith slot".) Now define

$$\partial c = \sum_{i=1}^{k} (-1)^i (c_{i,0} - c_{i,1}) = \sum_{i=1}^{k} \sum_{\rho=0,1} (-1)^{i+\rho} c_{i,\rho}.$$

For an arbitrary k-chain $c = \sum_i a_i c_i$ we put $\partial c = \sum_i a_i \partial c_i$. Then ∂ is a linear map from k-chains to $k+1$-chains.

There are a number of curious similarities between the boundary operator ∂ and the exterior derivative d, the most important of which is the following. (There are also many differences, such as the fact that d *raises* the degree of a form by 1, whereas ∂ *lowers* the dimension of a chain by 1.)

7.3. PROPOSITION. *$\partial(\partial c) = 0$ for every k-chain c in U. In short,*

$$\boxed{\partial^2 = 0.}$$

PROOF. By linearity of ∂ it suffices to prove this for k-cubes $c\colon [0,1]^k \to U$. Let $\mathbf{t} = (t_1, t_2, \dots, t_{k-2}) \in [0,1]^{k-2}$ and let ρ and τ be 0 or 1. Then for $1 \le i \le j \le k$ we have

$$\begin{aligned}(c_{i,\rho})_{j,\tau} &= c_{i,\rho}(t_1, t_2, \dots, t_{j-1}, \tau, t_j, \dots, t_{k-2})\\ &= c(t_1, t_2, \dots, t_{i-1}, \rho, t_i, \dots, t_{j-1}, \tau, t_j, \dots, t_{k-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}(c_{j+1,\tau})_{i,\rho} &= c_{j+1,\tau}(t_1, t_2, \dots, t_{i-1}, \rho, t_i, \dots, t_{k-2})\\ &= c(t_1, t_2, \dots, t_{i-1}, \rho, t_i, \dots, t_{j-1}, \tau, t_j, \dots, t_{k-2}),\end{aligned}$$

because in the vector $(t_1, t_2, \dots, t_{i-1}, \rho, t_i, \dots, t_{k-2})$ the entry t_j occupies the $j+1$st slot! We conclude that $(c_{i,\rho})_{j,\tau} = (c_{j+1,\tau})_{i,\rho}$ for $1 \le i \le j \le k$. Therefore the $k-2$-chain $\partial(\partial c)$ is given by

$$\begin{aligned}\partial(\partial c) &= \partial \sum_{i=1}^{k} \sum_{\rho=0,1} (-1)^{i+\rho} c_{i,\rho}\\ &= \sum_{i=1}^{k} \sum_{\rho=0,1} (-1)^{i+\rho} \partial c_{i,\rho}\\ &= \sum_{i=1}^{k} \sum_{j=1}^{k-1} \sum_{\rho,\tau=0,1} (-1)^{i+j+\rho+\tau} (c_{i,\rho})_{j,\tau}\\ &= 0,\end{aligned}$$

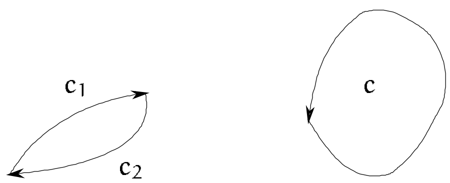
because in the sum each term $(c_{i,\rho})_{j,\tau} = (c_{j+1,\tau})_{i,\rho}$ occurs twice, once with sign $(-1)^{i+j+\rho+\tau}$ and once with the opposite sign $(-1)^{i+j+1+\rho+\tau}$. QED

A k-chain c is *closed*, or a *cycle*, if $\partial c = 0$. It is a *boundary* if $c = \partial b$ for some $k+1$-chain b.

7.4. COROLLARY. *Every boundary is a cycle.*

PROOF. If $c = \partial b$, then $\partial c = \partial(\partial b) = 0$. QED

7.5. EXAMPLE. If c_1 and c_2 are curves arranged head to tail as in the picture below, then $c_1 + c_2$ is a closed 1-chain. Likewise, any closed curve is a closed 1-chain.



In the same way that a closed form is not necessarily exact, it may happen that a closed 1-chain is not a boundary. See Example 7.7.

3. Stokes' theorem

In the language of chains and boundaries we can reformulate the fundamental theorem of calculus as follows:

$$\begin{aligned}\int_c dg &= g(c(1)) - g(c(0)) \\ &= \int_{c(1)} g - \int_{c(0)} g \\ &= \int_{c(1)-c(0)} g \\ &= \int_{\partial c} g.\end{aligned}$$

This is the form in which the fundamental theorem of calculus generalizes to higher dimensions. This generalization is perhaps the neatest relationship between the exterior derivative and the boundary operator. It contains as special cases the classical integration formulas of vector calculus (Green, Gauß and Stokes) and for that reason has Stokes' name attached to it, although it would perhaps be better to call it the "fundamental theorem of multivariable calculus".

7.6. THEOREM (Stokes' theorem). *Let α be a $k-1$-form on an open subset U of $\mathbf{R}^n$ and let c be a k-chain in U. Then*

$$\boxed{\int_c d\alpha = \int_{\partial c} \alpha.}$$

PROOF. By Theorem 5.11 we have

$$\int_c d\alpha = \int_{[0,1]^k} c^* d\alpha = \int_{[0,1]^k} dc^*\alpha.$$

Since $c^*\alpha$ is a $k-1$-form on $[0,1]^k$, it looks like

$$c^*\alpha = \sum_{i=1}^{k} g_i \, dt_1 \cdots \widehat{dt_i} \cdots dt_k.$$

Therefore

$$\begin{aligned}
\int_c d\alpha &= \sum_{i=1}^{k} \int_{[0,1]^k} d\big(g_i\, dt_1 \cdots \widehat{dt_i} \cdots dt_k\big) \\
&= \sum_{i=1}^{k} (-1)^{i+1} \int_{[0,1]^k} \frac{\partial g_i}{\partial t_i}\, dt_1 \cdots dt_k \\
&= \sum_{i=1}^{k} (-1)^{i+1} \int_{[0,1]^k} \frac{\partial g_i}{\partial t_i}\, dt_i\, dt_1 \cdots \widehat{dt_i} \cdots dt_k \qquad (*) \\
&= \sum_{i=1}^{k} (-1)^{i+1} \int_{[0,1]^{k-1}} \big(g_i(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_k) \\
&\qquad\qquad - g_i(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_k)\big)\, dt_1 \cdots \widehat{dt_i} \cdots dt_k \qquad (\dagger) \\
&= \sum_{i=1}^{k} (-1)^{i+1} \int_{[0,1]^{k-1}} (c_{i,1}^* \alpha - c_{i,0}^* \alpha) \\
&= \sum_{i=1}^{k} \sum_{\rho=0,1} (-1)^{i+\rho} \int_{[0,1]^{k-1}} c_{i,\rho}^* \alpha \\
&= \sum_{i=1}^{k} \sum_{\rho=0,1} (-1)^{i+\rho} \int_{c_{i,\rho}} \alpha \\
&= \int_{\partial c} \alpha.
\end{aligned}$$

Here in (*) we changed the order of integration and in (†) we applied the fundamental theorem of calculus in one variable. QED

7.7. EXAMPLE. The unit circle $c(t) = (\cos 2\pi t, \sin 2\pi t)$ is a 1-cycle in the punctured plane $U = \mathbf{R}^2 - \{0\}$. However, we claim that it is not a boundary in the sense that there exists no 2-chain b *contained in* U such that $c = \partial b$. This is intuitively obvious from the presence of the puncture in U, but using Stokes' theorem we can give an incontrovertible proof. Indeed, suppose that $c = \partial b$ for a 2-chain b in U. Let $\alpha = (-y\,dx + x\,dy)/(x^2 + y^2)$. Then on one hand

$$\int_c \alpha = \int_{\partial b} \alpha = \int_b d\alpha = 0,$$

where we have used Stokes' theorem and the fact that α is closed. On the other hand, the left-hand side is 2π times the winding number of the circle about the origin, and so $\int_c \alpha = 2\pi$, which is a contradiction. The moral of this example is that the presence of the puncture in U is responsible both for the existence of the non-exact closed 1-form α and for the closed 1-chain c which is not a boundary. Both phenomena were detected by using Stokes' formula.

Exercises

7.1. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^m$ and $\phi\colon U \to V$ a smooth map. Let c be a k-cube in U and α a k-form on V. Prove that $\int_c \phi^*\alpha = \int_{\phi\circ c} \alpha$.

7.2. Consider the curve $\mathbf{c}\colon [0,\pi/2] \to \mathbf{R}^2$ defined by $\mathbf{c}(t) = (a\cos t, b\sin t)$, where a and b are positive constants. Let $\alpha = xy\,dx + x^2y\,dy$.

(i) Sketch the curve $\mathbf{c}$ for $a = 2$ and $b = 1$.
(ii) Find $\int_{\mathbf{c}} \alpha$ (for arbitrary a and b).

7.3. Define a 2-cube $\mathbf{c}\colon [0,1]^2 \to \mathbf{R}^3$ by $\mathbf{c}(t_1,t_2) = \left(t_1^2, t_1t_2, t_2^2\right)$, and let $\alpha = x_1\,dx_2 + x_1\,dx_3 + x_2\,dx_3$.

(i) Sketch the image of $\mathbf{c}$.
(ii) Calculate both $\int_{\mathbf{c}} d\alpha$ and $\int_{\partial\mathbf{c}} \alpha$ and check that they are equal.

7.4. Define a 3-cube $\mathbf{c}\colon [0,1]^3 \to \mathbf{R}^3$ by $\mathbf{c}(t_1,t_2,t_3) = (t_2t_3, t_1t_3, t_1t_2)$, and let $\alpha = x_1\,dx_2\,dx_3$. Calculate both $\int_{\mathbf{c}} d\alpha$ and $\int_{\partial\mathbf{c}} \alpha$ and check that they are equal.

7.5. Let $R_1 \geq R_2 \geq 0$ be constants. Define a 3-cube $\mathbf{c}\colon [0,R_2] \times [0,2\pi] \times [0,2\pi] \to \mathbf{R}^3$ by

$$\mathbf{c}(r,\theta_1,\theta_2) = \big((R_1 + r\cos\theta_2)\cos\theta_1, (R_1 + r\cos\theta_2)\sin\theta_1, r\sin\theta_2\big).$$

(i) Sketch the image of $\mathbf{c}$.
(ii) Let x_1, x_2, x_3 be the Cartesian coordinates on $\mathbf{R}^3$. Compute $\mathbf{c}^*dx_1$, $\mathbf{c}^*dx_2$, $\mathbf{c}^*dx_3$ and $\mathbf{c}^*(dx_1\,dx_2\,dx_3)$.
(iii) Find the volume of the solid parametrized by $\mathbf{c}$.

7.6. Using polar coordinates in four dimensions (cf. Exercise 2.10) write the three-dimensional unit sphere S^3 in $\mathbf{R}^4$ as the image of a 3-cube $\mathbf{c}$. Calculate the boundary $\partial\mathbf{c}$ of this cube.

7.7. Deduce the following classical integration formulas from the generalized version of Stokes' theorem. All functions, vector fields, chains etc. are smooth and are defined in an open subset U of $\mathbf{R}^n$. (Some formulas hold only for special values of n, as indicated.)

(i) $\displaystyle\int_{\mathbf{c}} \operatorname{grad} g \cdot d\mathbf{x} = g(\mathbf{c}(1)) - g(\mathbf{c}(0))$ for any function g and any curve $\mathbf{c}$.
(ii) Green's formula: $\displaystyle\int_{\mathbf{c}} \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big) = \int_{\partial\mathbf{c}} (f\,dx + g\,dy)$ for any functions f, g and any 2-chain $\mathbf{c}$. (Here $n = 2$.)
(iii) Gauß' formula: $\displaystyle\int_{\mathbf{c}} \operatorname{div} F\,dx_1 \cdots dx_n = \int_{\partial\mathbf{c}} F \cdot d\mathbf{x}^*$ for any vector field F and any n-chain $\mathbf{c}$.
(iv) Stokes' formula: $\displaystyle\int_{\mathbf{c}} \operatorname{curl} F \cdot d\mathbf{x}^* = \int_{\partial\mathbf{c}} F \cdot d\mathbf{x}$ for any vector field F and any 2-chain $\mathbf{c}$. (Here $n = 3$.)

In parts (iii) and (iv) we use the formal notation

$$d\mathbf{x} = (dx_1, \dots, dx_n) \qquad \text{and} \qquad d\mathbf{x}^* = (dx_1^*, \dots, dx_n^*),$$

and the dot denotes inner product. We shall give a geometric interpretation of the entity $d\mathbf{x}^*$ in terms of volume forms later on. (See the discussion following Lemma 8.39.)