CHAPTER 8

Forms on manifolds

1. What is a differential form really?

Covectors. Before discussing manifolds and differential forms on them we need to be more precise about the definition of a differential form on $\mathbf{R}^n$. Recall that $\mathbf{R}^n$ is the collection of all column vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Let U be an open subset of $\mathbf{R}^n$. The definition of a 0-form on U does not need to be clarified: it is simply a function on U. Formally, a 1-form α on U can be defined as a *row vector*

$$\alpha = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix}$$

whose entries are functions on U. The form is called *constant* if the entries $f_1, \dots,$ f_n are constant. The set of constant row vectors is denoted by $(\mathbf{R}^n)^*$ and is called the *dual* of $\mathbf{R}^n$. Constant 1-forms are also known as *covariant vectors* or *covectors* and arbitrary 1-forms as *covariant vector fields* or *covector fields*. By definition dx_i is the constant 1-form

$$dx_i = (\mathbf{e}_i)^T = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix},$$

the transpose of $\mathbf{e}_i$, the ith standard basis vector of $\mathbf{R}^n$. Every 1-form can thus be written as

$$\alpha = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix} = \sum_{i=1}^n f_i\, dx_i.$$

Using this formalism we can write for any smooth function g on U

$$dg = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix},$$

so dg is simply the Jacobi matrix of g!

The notions of a covector and a 1-form make sense for any vector space. To see how, let us start by observing that a row vector $\mathbf{y}$ is nothing but a $1 \times n$-matrix. We can multiply it by a column vector $\mathbf{x}$ to obtain a number,

$$\mathbf{yx} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n y_i x_i.$$

Obviously we have $\mathbf{y}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{y}\mathbf{x}_1 + c_2\mathbf{y}\mathbf{x}_2$. Thus a row vector can be viewed as a linear map which sends column vectors in $\mathbf{R}^n$ to 1-dimensional vectors (scalars) in $\mathbf{R}^1 = \mathbf{R}$.

This motivates the following definition. If V is any (real) vector space (for example $\mathbf{R}^n$ or a subspace of $\mathbf{R}^n$), then V^*, the *dual* of V, is the set of linear maps from V to $\mathbf{R}$. Elements of V^* are called *covectors* or *linear functionals*. The dual is a vector space in its own right: if μ_1 and μ_2 are in V^* we define $\mu_1 + \mu_2$ and $c\mu_1$ by setting $(\mu_1 + \mu_2)(v) = \mu_1(v) + \mu_2(v)$ for all $v \in V$ and $(c\mu_1)(v) = c\mu_1(v)$.

8.1. Example. Let $V = C^0([a, b], \mathbf{R})$, the collection of all continuous real-valued functions on a closed and bounded interval $[a, b]$. A linear combination of continuous functions is continuous, so V is a vector space. Define $\mu(f) = \int_a^b f(x)\,dx$. Then $\mu(c_1f_1 + c_2f_2) = c_1\mu(f_1) + c_2\mu(f_2)$, so μ is a linear functional on V.

8.2. Example. Let $V = \mathbf{R}^n$ and fix $\mathbf{v} \in V$. Define $\mu(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$, where "$\cdot$" is the standard inner product on $\mathbf{R}^n$. Then μ is a linear functional on V.

Now suppose that V is a vector space of finite dimension n and choose a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ of V. Then every vector $\mathbf{v} \in V$ can be written in a unique way as a linear combination $\sum_j c_j\mathbf{v}_j$. Define a covector $\lambda_i \in V^*$ by $\lambda_i(\mathbf{v}) = c_i$. In other words, λ_i is determined by the rule

$$\lambda_i(\mathbf{v}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

8.3. Lemma. *The functions* $\lambda_1, \lambda_2, \dots, \lambda_n$ *form a basis of* V^*. *Hence* $\dim V^* = n = \dim V$.

Proof. Let $\lambda \in V^*$. We need to write λ as a linear combination $\lambda = \sum_{i=1}^n c_i\lambda_i$. Assuming for the moment that this is possible, we can apply both sides to the vector $\mathbf{v}_j$ to obtain

$$\lambda(\mathbf{v}_j) = \sum_{i=1}^n c_i\lambda_i(\mathbf{v}_j) = \sum_{i=1}^n c_i\delta_{ij} = c_j. \tag{8.1}$$

So $c_j = \lambda(\mathbf{v}_j)$ is the only possible choice for the coefficient c_j. To show that this choice of coefficients works, let us define $\lambda' = \sum_{i=1}^n \lambda(\mathbf{v}_i)\lambda_i$. Then by (8.1), $\lambda'(\mathbf{v}_j) = \lambda(\mathbf{v}_j)$ for all j, so $\lambda' = \lambda$, i.e. $\lambda = \sum_{i=1}^n \lambda(\mathbf{v}_i)\lambda_i$. We have proved that every $\lambda \in V^*$ can be written uniquely as a linear combination of the λ_i. QED

The basis $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of V^* is said to be *dual* to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of V.

8.4. Example. Consider $\mathbf{R}^n$ with standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then $dx_i(\mathbf{e}_j) = \mathbf{e}_i^T\mathbf{e}_j = \delta_{ij}$, so the dual basis of $(\mathbf{R}^n)^*$ is $\{dx_1, dx_2, \dots, dx_n\}$.

Multilinear algebra. Let V be a vector space and let V^k denote the product $V \times \cdots \times V$ (k times). Thus an element of V^k is an ordered k-tuple $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ of vectors in V. A k-*linear* function (also called a *multilinear* function) on V is a function $\lambda\colon V^k \to \mathbf{R}$ which is linear in each vector, i.e.

$$\begin{aligned} &\lambda(\mathbf{v}_1, \mathbf{v}_2, \dots, c\mathbf{v}_i + c'\mathbf{v}_i', \dots, \mathbf{v}_k) \\ &\qquad = c\lambda(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) + c'\lambda(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_i', \dots, \mathbf{v}_k) \end{aligned}$$

for all scalars c, c' and all vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_i, \mathbf{v}_i', \dots, \mathbf{v}_k$.

8.5. EXAMPLE. Let $V = \mathbf{R}^n$ and let $\lambda(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, the dot product of $\mathbf{x}$ and $\mathbf{y}$. Then λ is bilinear (i.e. 2-linear).

8.6. EXAMPLE. Let $V = \mathbf{R}^4$, $k = 2$. The function $\lambda(\mathbf{v}, \mathbf{w}) = v_1w_2 - v_2w_1 + v_3w_4 - v_4w_3$ is bilinear on $\mathbf{R}^4$.

8.7. EXAMPLE. Let $V = \mathbf{R}^n$, $k = n$. The determinant $\det(\mathbf{v}_1\mathbf{v}_2,, \ldots, \mathbf{v}_n)$ is an n-linear function on $\mathbf{R}^n$.

A k-linear function is *alternating* or *antisymmetric* if

$$\lambda(\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k) = -\lambda(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k).$$

More generally, if λ is alternating, then for any permutation $\sigma \in S_k$ we have

$$\lambda(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}) = \operatorname{sign}(\sigma)\lambda(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

8.8. EXAMPLE. The dot product of Example 8.5 is bilinear, but it is not alternating. Indeed it is *symmetric*: $\mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. The bilinear function of Example 8.6 is alternating, and so is the determinant function of Example 8.7.

Here is a useful trick to generate alternating k-linear functions starting from k covectors $\lambda_1, \lambda_2, \ldots, \lambda_k \in V^*$. The *(wedge) product* is the function

$$\lambda_1\lambda_2 \cdots \lambda_k \colon V^k \to \mathbf{R}$$

defined by

$$\boxed{\lambda_1\lambda_2 \cdots \lambda_k(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \det\big(\lambda_i(\mathbf{v}_j)\big)_{1 \leq i,j \leq k}.}$$

(The determinant on the right is a $k \times k$-determinant.) It follows from the multilinearity and the alternating property of the determinant that $\lambda_1\lambda_2 \cdots \lambda_k$ is an alternating k-linear function. The wedge product is often denoted by $\lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_k$ to distinguish it from other products, such as the tensor product defined in Problem 8.6.

The collection of all alternating k-linear functions is denoted by A^kV.

For $k = 1$ the alternating property is vacuous, so an alternating 1-linear function is nothing but a linear function. Thus $A^1V = V^*$.

For $k = 0$ a k-linear function is defined to be a single number. Thus $A^0V = \mathbf{R}$.

For any k, k-linear functions can be added and scalar-multiplied just like ordinary linear functions, so the set A^kV forms a vector space.

There is a nice way to construct a basis of the vector space A^kV starting from a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V. The idea is to take wedge products of dual basis vectors. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the corresponding dual basis of V^*. Let $I = (i_1, i_2, \ldots, i_k)$ be an increasing multi-index, i.e. $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Write

$$\lambda_I = \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_k} \in A^kV,$$
$$\mathbf{v}_I = (\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}) \in V^k.$$

8.9. EXAMPLE. Let $V = \mathbf{R}^3$ with standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The dual basis of $(\mathbf{R}^3)^*$ is $\{dx_1, dx_2, dx_3\}$. Let $k = 2$ and $I = (1,2)$, $J = (2,3)$. Then

$$dx_I(\mathbf{e}_I) = \begin{vmatrix} dx_1(\mathbf{e}_1) & dx_1(\mathbf{e}_2) \\ dx_2(\mathbf{e}_1) & dx_2(\mathbf{e}_2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$dx_I(\mathbf{e}_J) = \begin{vmatrix} dx_1(\mathbf{e}_2) & dx_1(\mathbf{e}_3) \\ dx_2(\mathbf{e}_2) & dx_2(\mathbf{e}_3) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0,$$

$$dx_J(\mathbf{e}_I) = \begin{vmatrix} dx_2(\mathbf{e}_1) & dx_2(\mathbf{e}_2) \\ dx_3(\mathbf{e}_1) & dx_3(\mathbf{e}_2) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0,$$

$$dx_J(\mathbf{e}_J) = \begin{vmatrix} dx_2(\mathbf{e}_2) & dx_2(\mathbf{e}_3) \\ dx_3(\mathbf{e}_2) & dx_3(\mathbf{e}_3) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

This example generalizes as follows.

8.10. LEMMA. *Let I and J be increasing multi-indices of degree k. Then*

$$\lambda_I(\mathbf{v}_J) = \delta_{IJ} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

PROOF. Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$. Then

$$\lambda_I(\mathbf{v}_J) = \det\big(\lambda_{i_r}(\mathbf{v}_{j_s})\big)_{1 \le r,s \le k} = \begin{vmatrix} \delta_{i_1,j_1} & \dots & \delta_{i_1,j_k} \\ \vdots & & \vdots \\ \delta_{i_l,j_1} & \dots & \delta_{i_l,j_k} \\ \vdots & & \vdots \\ \delta_{i_k,j_1} & \dots & \delta_{i_k,j_k} \end{vmatrix} = 1 \quad \text{if } I = J.$$

If $I \neq J$, then $i_l \neq j_l$ for some l. Choose l as small as possible, so that $i_m = j_m$ for $m < l$. There are two cases: $i_l < j_l$ and $i_l > j_l$. If $i_l < j_l$, then $i_l < j_l < j_{l+1} < \cdots < j_k$ because J is increasing, so all entries δ_{i_l,j_m} in the determinant with $m \geq l$ are 0. For $m < l$ we have $j_m = i_m < i_l$ because I is increasing, so $\delta_{i_l,j_m} = 0$ for $m < l$. In other words the lth row in the determinant is 0 and hence $\lambda_I(\mathbf{v}_J) = 0$. If $i_l > j_l$ we find that the lth column in the determinant is 0 and therefore again $\lambda_I(\mathbf{v}_J) = 0$. QED

We need one further technical result before showing that the functions λ_I form a basis of A^kV.

8.11. LEMMA. *Let $\lambda \in A^kV$. Suppose $\lambda(\mathbf{v}_I) = 0$ for all increasing multi-indices I of degree k. Then $\lambda = 0$.*

PROOF. The assumption implies

$$\lambda(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_k}) = 0 \tag{8.2}$$

for *all* multi-indices $(i_1, \dots, i_k)$, because of the alternating property. We need to show that $\lambda(\mathbf{w}_1, \dots, \mathbf{w}_k) = 0$ for arbitrary vectors $\mathbf{w}_1, \dots, \mathbf{w}_k$. We can expand the $\mathbf{w}_i$ using the basis:

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + \cdots + a_{1k}\mathbf{v}_k, \\ &\vdots \\ \mathbf{w}_k &= a_{k1}\mathbf{v}_1 + \cdots + a_{kk}\mathbf{v}_k. \end{aligned}$$

Therefore by multilinearity

$$\lambda(\mathbf{w}_1, \dots, \mathbf{w}_k) = \sum_{i_1=1}^{k} \cdots \sum_{i_k=1}^{k} a_{1i_1} \cdots a_{ki_k} \lambda(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_k}).$$

Each term in the right-hand side is 0 by (8.2). QED

8.12. THEOREM. *Let V be an n-dimensional vector space with basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the corresponding dual basis of V^*. Then the alternating functions $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_k}$, where I ranges over the set of all increasing multi-indices of degree k, form a basis of A^kV. Hence* $\dim A^kV = \binom{n}{k}$.

PROOF. The proof is closely analogous to that of Lemma 8.3. Let $\lambda \in A^kV$. We need to write λ as a linear combination $\lambda = \sum_I c_I \lambda_I$. Assuming for the moment that this is possible, we can apply both sides to the k-tuple of vectors $\mathbf{v}_J$. Using Lemma 8.10 we obtain

$$\lambda(\mathbf{v}_J) = \sum_I c_I \lambda_I(\mathbf{v}_J) = \sum_I c_I \delta_{IJ} = c_J.$$

So $c_J = \lambda(\mathbf{v}_J)$ is the only possible choice for the coefficient c_J. To show that this choice of coefficients works, let us define $\lambda' = \sum_I \lambda(\mathbf{v}_I)\lambda_I$. Then for all increasing multi-indices I we have $\lambda(\mathbf{v}_I) - \lambda'(\mathbf{v}_I) = \lambda(\mathbf{v}_I) - \lambda(\mathbf{v}_I) = 0$. Applying Lemma 8.11 to $\lambda - \lambda'$ we find $\lambda - \lambda' = 0$. In other words, $\lambda = \sum_I \lambda(\mathbf{v}_I)\lambda_I$. We have proved that every $\lambda \in V^*$ can be written uniquely as a linear combination of the λ_i. QED

8.13. EXAMPLE. Let $V = \mathbf{R}^n$ with standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. The dual basis of $(\mathbf{R}^n)^*$ is $\{dx_1, \dots, dx_n\}$. Therefore A^kV has a basis consisting of all k-linear functions of the form

$$dx_I = dx_{i_1} \cdots dx_{i_k},$$

with $1 \le i_1 < \cdots < i_k \le n$. Hence a general alternating k-linear function λ on $\mathbf{R}^n$ looks like

$$\lambda = \sum_I a_I dx_I,$$

with a_I constant. The value of the constant is $a_I = \lambda(\mathbf{e}_I)$.

An arbitary k-form α on a region U in $\mathbf{R}^n$ is now defined as a choice of an alternating k-linear function $\alpha_\mathbf{x}$ for each $\mathbf{x} \in U$; hence it looks like $\alpha_\mathbf{x} = \sum_I f_I(\mathbf{x}) dx_I$, where the coefficients f_I are *functions* on U. We shall abbreviate this to

$$\alpha = \sum_I f_I dx_I,$$

and we shall always assume the coefficients to be smooth functions. This is a rigorous, unequivocal definition of a differential form on $\mathbf{R}^n$,

Pullbacks re-examined. In terms of this new definition we can give a fresh interpretation of a pullback. This will be useful in our study of forms on manifolds. Let us start in the setting of a pair of abstract vector spaces V and W and a linear map $L\colon V \to W$. Consider an alternating k-linear function $\lambda \in A^kW$. The *pullback* of λ under L is the function $L^*\lambda\colon V^k \to \mathbf{R}$ defined by

$$L^*\lambda(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \lambda(L\mathbf{v}_1, L\mathbf{v}_2, \dots, L\mathbf{v}_k).$$

Then $L^*\lambda$ is k-linear and alternating, so $L^*\lambda \in A^kV$. Thus the pullback operation can be regarded as a map $L^*\colon A^kW \to A^kV$.

Let us make this more explicit for $V = \mathbf{R}^n$ with standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ and $W = \mathbf{R}^m$ with standard basis $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$. Then we have $L\mathbf{e}_j = \sum_{i=1}^m l_{ij}\mathbf{f}_i$, where the l_{ij} are the entries of the $m \times n$-matrix of L. We can now compute L^*dy_I. Plugging in $\mathbf{e}_J$ we get

$$\begin{aligned}(8.3)\qquad L^*dy_I(\mathbf{e}_J) = dy_I(L\mathbf{e}_{j_1}, L\mathbf{e}_{j_2}, \dots, L\mathbf{e}_{j_k}) &= \det(dy_{i_p}(L\mathbf{e}_{j_q}))_{1\le p,q\le k}\\ &= \det(l_{i_p,j_q})_{1\le p,q\le k} = \det L_{IJ},\end{aligned}$$

and so $L^*dy_I = \sum_J \det L_{IJ}\,dx_J$. More generally, if $\lambda = \sum_I a_I dy_I$, then

$$L^*\lambda = \sum_I a_I L^*dy_I = \sum_I\sum_J a_I \det L_{IJ}\,dx_J = \sum_J b_J dx_J,$$

where $b_J = \sum_I a_I \det L_{IJ}$.

Now let U and V be open subsets of $\mathbf{R}^n$, resp. $\mathbf{R}^m$, and $\phi\colon U \to V$ a smooth map. For a k-form $\alpha \in \Omega^k(V)$ define the pullback $\phi^*\alpha \in \Omega^k(U)$ by

$$\boxed{(\phi^*\alpha)_{\mathbf{x}}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \alpha_{\phi(\mathbf{x})}(d\phi(\mathbf{x})\mathbf{v}_1, d\phi(\mathbf{x})\mathbf{v}_2, \dots, d\phi(\mathbf{x})\mathbf{v}_k).}$$

Let us check that this formula agrees with the old definition. Following the same line of thought as in (8.3) we find $\phi^*dy_I = \sum_J \det d\phi_{IJ}\,dx_J$. Hence if $\alpha = \sum_I f_I dy_I$,

$$(\phi^*\alpha)_{\mathbf{x}} = \sum_I f(\phi(\mathbf{x}))\phi^*dy_I = \sum_I f(\phi(\mathbf{x}))\sum_J \det d\phi(\mathbf{x})_{IJ}\,dx_J = \sum_J g_J(\mathbf{x})dx_J,$$

where $g_J = \sum_I \phi^*f_I \det d\phi_{IJ}$. This formula is identical to the one in Theorem 5.8 and therefore our new definition agrees with the old!

2. Forms on manifolds

Let M be an n-dimensional manifold in $\mathbf{R}^N$. We know that for each $\mathbf{p}$ in M the tangent space $T_{\mathbf{p}}M$ is a k-dimensional linear subspace of $\mathbf{R}^N$. A *differential form of degree* k, or simply a k-*form*, α on M is a choice of an alternating k-linear map $\alpha_{\mathbf{p}}$ on the vector space $T_{\mathbf{p}}M$, one for each $\mathbf{p} \in M$.

8.14. EXAMPLE. Let M be a one-dimensional manifold in $\mathbf{R}^N$. Let us choose an orientation ("direction") on M. A tangent vector to M is *positive* if it points in the same direction as the orientation and *negative* if it points in the opposite direction. Define a 1-form α on M as follows. For $\mathbf{p} \in M$ and a tangent vector $\mathbf{v} \in T_{\mathbf{p}}M$ put

$$\alpha_{\mathbf{p}}(\mathbf{v}) = \begin{cases} \|\mathbf{v}\| & \text{if } \mathbf{v} \text{ is positive,}\\ -\|\mathbf{v}\| & \text{if } \mathbf{v} \text{ is negative.}\end{cases}$$

This form is the *element of arclength* of M. We shall see in the next chapter how to use it to calculate the length of M and also how to generalize it to higher-dimensional manifolds.

We can give a more concrete description of a form on a manifold as follows. Choose $\mathbf{p}$ in M. According to the definition of a manifold there exist an open set U in $\mathbf{R}^n$ containing $\mathbf{p}$ and a map $\Phi\colon U \to \mathbf{R}^N$ such that $\Phi(U)$ is open, $\Phi\colon U \to \Phi(U)$ has a smooth inverse, and $M \cap U = \Phi^{-1}(\mathbf{R}^n)$. Let us put $W = M \cap U$ and $V = \Phi(U) \cap \mathbf{R}^k$. Then V is open in $\mathbf{R}^k$ and Φ maps W bijectively onto V. Let us write $\phi\colon W \to V$ for Φ considered as a map from W to V, and $\psi\colon V \to W$ for the inverse of ϕ. ϕ is called a *chart* or *local coordinate system* on M and ψ is a *local parametrization* of M. Finally recall that the tangent space of $T_{\mathbf{p}}M$ is defined by

$$T_{\mathbf{p}}M = \big(d\Phi(\mathbf{p})\big)^{-1}(\mathbf{R}^n) = d\psi(\phi(\mathbf{p}))(\mathbf{R}^n).$$

If α is a k-form on M, we define its *pullback* under the local parametrization ψ by

$$(\psi^*\alpha)_{\mathbf{x}}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \alpha_{\psi(\mathbf{x})}\big(d\psi(\mathbf{x})\mathbf{v}_1, d\psi(\mathbf{x})\mathbf{v}_2, \dots, d\psi(\mathbf{x})\mathbf{v}_k\big).$$

Then $\psi^*\alpha$ is a k-form on V, an open subset of $\mathbf{R}^n$, so $\psi^*\alpha = \sum_I f_I\,dx_I$ for certain functions f_I defined on V. Often we say, rather sloppily, that "$\alpha = \sum_I f_I\,dx_I$ in the chart ϕ".

We can calculate $\psi^*\alpha \in \Omega^k(W)$ for any local parametrization $\psi\colon W \to M$ of M. Suppose we have two different charts $\phi_1\colon W_1 \to \mathbf{R}^n$ and $\phi_2\colon W_2 \to \mathbf{R}^n$, with corresponding inverses $\psi_1\colon \phi_1(W_1) \to \phi_2(W_2)$, such that $\mathbf{x}$ is contained in both W_1 and W_2. How do the local expressions $\alpha_1 = \psi_1^*\alpha$ and $\alpha_2 = \psi_2^*\alpha$ for α compare? To answer this question, consider the map

$$\phi_2 \circ \phi_1^{-1} = \psi_2^{-1} \circ \psi_1,$$

which maps $\phi_1(W_1 \cap W_2)$ to $\phi_2(W_1 \cap W_2)$. This map is called the *coordinate change*. From $\alpha_1 = \psi_1^*\alpha$ and $\alpha_2 = \psi_2^*\alpha$ we get

$$\boxed{(\psi_2^{-1} \circ \psi_1)^*\alpha_2 = \alpha_1.} \tag{8.4}$$

This gives an alternative view of a differential form on M: α can be regarded as a collection of forms $\alpha_\psi \in \Omega^k(W)$, one for each local parametrization $\psi\colon W \to M$ of M, such that (8.4) holds for any pair of local parametrizations $\psi_1\colon W_1 \to M$ and $\psi_2\colon W_2 \to M$.

Sometimes it is useful to write (8.4) in components. We can do this by appealing to Theorem 5.8. If

$$\alpha_1 = \sum_I f_I\,dx_I \qquad \text{and} \qquad \alpha_2 = \sum_J f_J\,dx_J$$

are the local expressions for α, then

$$g_J = \sum_I (\phi_2 \circ \phi_1^{-1})^* f_I \ \det d(\phi_2 \circ \phi_1^{-1})_{IJ}.$$

Exercises

8.1. The vectors $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 - \mathbf{e}_2$ form a basis of $\mathbf{R}^2$. What is the dual basis of $(\mathbf{R}^2)^*$?

8.2. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of $\mathbf{R}^n$ and let $\{\lambda_1, \dots, \lambda_n\}$ be the dual basis of $(\mathbf{R}^n)^*$. Let A be an invertible $n \times n$-matrix. Then by elementary linear algebra the set of vectors $\{A\mathbf{v}_1, \dots, A\mathbf{v}_n\}$ is also a basis of $\mathbf{R}^n$. Show that the corresponding dual basis is the set of row vectors $\{\lambda_1 A^{-1}, \dots, \lambda_n A^{-1}\}$.

8.3. Suppose that μ is a bilinear function on a vector space V satisfying $\mu(\mathbf{v}, \mathbf{v}) = 0$ for all vectors $\mathbf{v} \in V$. Prove that μ is alternating. Generalize this observation to k-linear functions.

8.4. Show that the bilinear function μ of Example 8.6 is equal to $dx_1\,dx_2 + dx_3\,dx_4$.

8.5. The wedge product is a generalization of the cross product to arbitrary dimensions in the sense that

$$\mathbf{x} \times \mathbf{y} = \big(*(\mathbf{x}^{\mathsf{T}} \wedge \mathbf{y}^{\mathsf{T}})\big)^{\mathsf{T}}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$. Prove this formula. (Interpretation: $\mathbf{x}$ and $\mathbf{y}$ are column vectors, $\mathbf{x}^{\mathsf{T}}$ and $\mathbf{y}^{\mathsf{T}}$ are row vectors, $\mathbf{x}^{\mathsf{T}} \wedge \mathbf{y}^{\mathsf{T}}$ is a 2-form on $\mathbf{R}^3$, $*(\mathbf{x}^{\mathsf{T}} \wedge \mathbf{y}^{\mathsf{T}})$ is a 1-form, i.e. a row vector. So both sides of the formula represent column vectors.)

8.6. Let V be a vector space and let $\mu_1, \mu_2, \dots, \mu_k \in V^*$ be covectors. Their *tensor product* is the function

$$\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_k \colon V^k \to \mathbf{R}$$

defined by

$$\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_k(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \mu_1(\mathbf{v}_1)\mu_2(\mathbf{v}_2) \cdots \mu_k(\mathbf{v}_k).$$

Show that $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_k$ is a k-linear function.

8.7. Let $\mu \colon V^k \to \mathbf{R}$ be a k-linear function. Define a new function $\operatorname{Alt}\mu \colon V^k \to \mathbf{R}$ by

$$\operatorname{Alt}\mu(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\mu(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}).$$

Prove the following.

(i) $\operatorname{Alt}\mu$ is an *alternating* k-linear function.
(ii) $\operatorname{Alt}\mu = \mu$ if μ is alternating.
(iii) $\operatorname{Alt}\operatorname{Alt}\mu = \operatorname{Alt}\mu$ for all k-linear μ.
(iv) Let $\mu_1, \mu_2, \dots, \mu_k \in V^*$. Then

$$\mu_1\mu_2 \cdots \mu_k = \frac{1}{k!}\operatorname{Alt}(\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_k).$$

8.8. Show that $\det(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = dx_1\,dx_2 \cdots dx_n(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ for all vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbf{R}^n$. In short,

$$\boxed{\det = dx_1\,dx_2 \cdots dx_n.}$$

8.9. Let V and W be vector spaces and $L \colon V \to W$ a linear map. Show that $L^*(\lambda\mu) = (L^*\lambda)(L^*\mu)$ for all covectors $\lambda, \mu \in W^*$.