CHAPTER 9

Volume

1. Orientations

Oriented vector spaces. You are probably familiar with orientations on vector spaces of dimension ≤ 3. An orientation of a line is an assignment of a direction. An orientation of a plane is a choice of a direction of rotation. An orientation of a three-dimensional space is a choice of "handedness", i.e. a right-hand rule versus a left-hand rule.

These notions can be generalized as follows. Let V be an n-dimensional (real) vector space. Suppose that $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ and $\mathcal{B}' = (\mathbf{v}'_1, \mathbf{v}'_2, \dots, \mathbf{v}'_n)$ are two ordered bases of V. Then we can write $\mathbf{v}'_i = \sum_j a_{ij}\mathbf{v}_j$ and $v_i = \sum_j b_{ij} v'_j$ for suitable coefficients a_{ij} and b_{ij}. The $n \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij})$ satisfy $AB = BA = I$ and are therefore invertible. We say that the bases $\mathcal{B}$ and $\mathcal{B}'$ define the *same orientation* of V if if $\det A > 0$. If $\det A < 0$, the two bases define *opposite orientations*.

For instance, if $(\mathbf{v}'_1, \mathbf{v}'_2, \dots, \mathbf{v}'_n) = (\mathbf{v}_2, \mathbf{v}_1, \dots, \mathbf{v}_n)$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

so $\det A = -1$. Hence the ordered bases $(\mathbf{v}_2, \mathbf{v}_1, \dots, \mathbf{v}_n)$ and $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ define opposite orientations. More generally, if $\sigma \in S_n$ is any permutation, then the ordered basis $(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(n)})$ defines the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ if $\operatorname{sign}\sigma = 1$, and the opposite orientation if $\operatorname{sign}\sigma = -1$.

We know now what it means for two bases to have the same orientation, but how do we define the concept of an orientation itself? In typical mathematician's fashion we define the orientation of V determined by the basis $\mathcal{B}$ to be the collection of all ordered bases that have the same orientation as $\mathcal{B}$. (There is a somewhat analogous definition of the number 1, namely as the collection of all sets that contain one element.) The orientation determined by $\mathcal{B}$ is denoted by $[\mathcal{B}]$ or $[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$. So if $\mathcal{B}$ and $\mathcal{B}'$ define the same orientation then $[\mathcal{B}] = [\mathcal{B}']$. If they define opposite orientations we write $[\mathcal{B}] = -[\mathcal{B}']$. Because the determinant of an invertible matrix is either positive or negative, there are two possible orientations of V. An *oriented vector space* is a vector space together with an arbitrary choice of an orientation. This preferred orientation is then called *positive*.

For $n = 0$ we need to make a special definition, because a 0-dimensional space has an empty basis. In this case we define an orientation of V to be a choice of sign, $+$ or $-$.

9.1. EXAMPLE. The *standard orientation* on $\mathbf{R}^n$ is the orientation $[\mathbf{e}_1, \dots, \mathbf{e}_n]$ defined by the standard ordered basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$. We shall always use this orientation on $\mathbf{R}^n$.

Maps and orientations. Let V and W be oriented vector spaces of the same dimension and let $L\colon V \to W$ be an invertible linear map. Choose a positively oriented basis $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ of V. Because L is invertible, the ordered n-tuple $(L\mathbf{v}_1, L\mathbf{v}_2, \dots, L\mathbf{v}_n)$ is an ordered basis of W. If this basis is positively, resp. negatively, oriented we say that L is *orientation-preserving*, resp. *orientation-reversing*. This definition does not depend on the choice of the basis, for if $(\mathbf{v}_1', \mathbf{v}_2', \dots, \mathbf{v}_n')$ is another positively oriented basis of V, then $v_i' = \sum_j a_{ij} v_j$ with $\det(a_{ij}) > 0$. Therefore $L\mathbf{v}_i' = L\bigl(\sum_j a_{ij}\mathbf{v}_j\bigr) = \sum_j a_{ij} L\mathbf{v}_j$, and hence the two bases $(L\mathbf{v}_1, L\mathbf{v}_2, \dots, L\mathbf{v}_n)$ and $(L\mathbf{v}_1', L\mathbf{v}_2', \dots, L\mathbf{v}_n')$ of W determine the same orientation.

Oriented manifolds. Now let M be a manifold. We define an *orientation* of M to be a choice of an orientation for each tangent space $T_{\mathbf{x}}M$ which varies continuously over M. "Continuous" means that for every $\mathbf{x} \in M$ there exists a local parametrization $\psi\colon W \to M$, with W open in $\mathbf{R}^n$ and $\mathbf{x} \in \psi(W)$, such that $d\phi_{\mathbf{y}}\colon \mathbf{R}^n \to T_{\mathbf{y}}M$ preserves the orientation for all $\mathbf{y} \in W$. A manifold is *orientable* if it possesses an orientation; it is *oriented* if a specific orientation has been chosen.

Hypersurfaces. The case of a hypersurface, a manifold of codimension 1, is particularly instructive. A *unit normal vector field* on a manifold M in $\mathbf{R}^n$ is a smooth function $\mathbf{n}\colon M \to S^{n-1}$ such that $\mathbf{n}(\mathbf{x}) \perp T_{\mathbf{x}}M$ for all $\mathbf{x} \in M$.

9.2. PROPOSITION. *A hypersurface in $\mathbf{R}^n$ is orientable if and only if it possesses a unit normal vector field.*

PROOF. Let M be a hypersurface in $\mathbf{R}^n$. Suppose M possesses a unit normal vector field. Let $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ be an ordered basis of $T_{\mathbf{x}}M$ for some $\mathbf{x} \in M$. Then $(\mathbf{n}(\mathbf{x}), \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ is a basis of $\mathbf{R}^n$, because $\mathbf{n}(\mathbf{x}) \perp \mathbf{v}_i$ for all i. We say that $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ is positively oriented if $(\mathbf{n}(\mathbf{x}), \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ is a positively oriented basis of $\mathbf{R}^n$. This defines an orientation on M, called the orientation *induced* by the normal vector field $\mathbf{n}$.

Conversely, let us suppose that M is an oriented hypersurface in $\mathbf{R}^n$. For each $\mathbf{x} \in M$ the tangent space $T_{\mathbf{x}}M$ is $n-1$-dimensional, so its orthogonal complement $(T_{\mathbf{x}}M)^{\perp}$ is a line. There are therefore precisely two vectors of length 1 which are perpendicular to $T_{\mathbf{x}}M$. We can pick a preferred unit normal vector as follows. Let $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ be a positively oriented basis of $T_{\mathbf{x}}M$. The *positive* unit normal vector is that unit normal vector $\mathbf{n}(\mathbf{x})$ that makes $(\mathbf{n}(\mathbf{x}), \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ a positively oriented basis of $\mathbf{R}^n$. In Exercise 9.1 you will be asked to check that $\mathbf{n}(\mathbf{x})$ depends smoothly on $\mathbf{x}$. In this way we have produced a unit normal vector field on M. QED

The existence of a unit normal on a hypersurface enables one to distinguish between two sides of M: the direction of $\mathbf{n}$ is "out" or "up"; the opposite direction is "in" or "down". For this reason orientable hypersurfaces are often called *two-sided*, whereas the nonorientable ones are called *one-sided*. Let us show that a hypersurface given by a single equation is always orientable.

9.3. PROPOSITION. *Let U be open in $\mathbf{R}^n$ and let $f\colon U \to \mathbf{R}$ be a smooth function. Let c be a regular value of f. Then the manifold $f^{-1}(c)$ has a unit normal vector field given by $\mathbf{n}(\mathbf{x}) = \operatorname{grad} f(\mathbf{x})/\|\operatorname{grad} f(\mathbf{x})\|$ and is therefore orientable.*

PROOF. The regular value theorem tells us that $M = f^{-1}(c)$ is a hypersurface in $\mathbf{R}^n$ (if nonempty), and also that $T_{\mathbf{x}}M = \ker df_{\mathbf{x}} = \big(\operatorname{grad} f(\mathbf{x})\big)^{\perp}$. The function $\mathbf{n}(\mathbf{x}) = \operatorname{grad} f(\mathbf{x})/\|\operatorname{grad} f(\mathbf{x})\|$ therefore defines a unit normal vector field on M. Appealing to Proposition 9.2 we conclude that M is orientable. QED

9.4. EXAMPLE. Taking $f(\mathbf{x}) = \|\mathbf{x}\|^2$ and $c = 1$ we obtain that the unit sphere S^{n-1} is orientable. The unit normal is $\mathbf{n}(\mathbf{x}) = \mathbf{x}$.

2. k-Dimensional volume in $\mathbf{R}^n$

Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be vectors in $\mathbf{R}^n$. The *block* or *parallelepiped* spanned by these vectors is the set of all vectors of the form $\sum_{i=1}^k c_i\mathbf{a}_i$, where the coefficients c_i range over the unit interval $[0, 1]$. For $k = 1$ this is also called a *line segment* and for $k = 2$ a *parallelogram*. We will need a formula for the volume of a block. There is no way of defining a natural order among the k vectors, so this volume will be an absolute volume. We approach this problem in a similar way as the problem of defining the determinant, namely by imposing a few reasonable axioms.

9.5. DEFINITION. An *absolute k-dimensional Euclidean volume function* is a function

$$\operatorname{vol}_k\colon \underbrace{\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{k \text{ times}} \to \mathbf{R}$$

with the following properties:

(i) homogeneity:

$$\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, c\mathbf{a}_i, \dots, \mathbf{a}_k) = |c| \operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k)$$

for all scalars c and all vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$;

(ii) invariance under shear transformations:

$$\operatorname{vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_i + c\mathbf{a}_j, \dots, \mathbf{a}_j, \dots, \mathbf{a}_k) = \operatorname{vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_j, \dots, \mathbf{a}_i, \dots, \mathbf{a}_k)$$

for all scalars c and any $i \neq j$;

(iii) invariance under Euclidean motions:

$$\operatorname{vol}_k(Q\mathbf{a}_1, \dots, Q\mathbf{a}_k) = \operatorname{vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)$$

for all orthogonal matrices Q;

(iv) normalization: $\operatorname{vol}_k(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k) = 1$.

We shall shortly see that these axioms uniquely determine the k-dimensional volume function.

9.6. LEMMA. (i) $\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = \|\mathbf{a}_1\|\,\|\mathbf{a}_2\| \cdots \|\mathbf{a}_k\|$ *if* $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ *are nonzero orthogonal vectors.*

(ii) $\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = 0$ *if the vectors* $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ *are dependent.*

PROOF. Suppose $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are orthogonal and nonzero. Define $\mathbf{q}_i = \|\mathbf{a}_i\|^{-1}\mathbf{a}_i$. Then the vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_k$ are ortho*normal*. Complete them to an orthonormal basis $\mathbf{q}_1, \mathbf{q}_2 \dots, \mathbf{q}_k, \mathbf{q}_{k+1}, \dots, \mathbf{q}_n$ of $\mathbf{R}^n$. Let Q be the matrix whose ith column is $\mathbf{q}_i$. Then Q is orthogonal and $Q\mathbf{e}_i = \mathbf{q}_i$. Therefore

$$\begin{aligned}
\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) &= \|\mathbf{a}_1\|\,\|\mathbf{a}_2\| \cdots \|\mathbf{a}_k\| \operatorname{vol}_k(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_k) && \text{by Axiom (i)}\\
&= \|\mathbf{a}_1\|\,\|\mathbf{a}_2\| \cdots \|\mathbf{a}_k\| \operatorname{vol}_k(Q\mathbf{e}_1, Q\mathbf{e}_2, \dots, Q\mathbf{e}_k)\\
&= \|\mathbf{a}_1\|\,\|\mathbf{a}_2\| \cdots \|\mathbf{a}_k\| \operatorname{vol}_k(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k) && \text{by Axiom (iii)}\\
&= \|\mathbf{a}_1\|\,\|\mathbf{a}_2\| \cdots \|\mathbf{a}_k\| && \text{by Axiom (iv),}
\end{aligned}$$

which proves (i).

Now assume $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are dependent. For simplicity suppose $\mathbf{a}_1$ is a linear combination of the other vectors, $\mathbf{a}_1 = \sum_{i=2}^k c_i \mathbf{a}_i$. By repeatedly applying Axiom (ii) we get

$$\begin{aligned}
\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) &= \operatorname{vol}_k\left(\sum_{i=2}^k c_i \mathbf{a}_i, \mathbf{a}_2, \dots, \mathbf{a}_k\right)\\
&= \operatorname{vol}_k\left(\sum_{i=3}^k c_i \mathbf{a}_i, \mathbf{a}_2, \dots, \mathbf{a}_k\right) = \cdots = \operatorname{vol}_k(\mathbf{0}, \mathbf{a}_2, \dots, \mathbf{a}_k).
\end{aligned}$$

Now by Axiom (i),

$$\operatorname{vol}_k(\mathbf{0}, \mathbf{a}_2, \dots, \mathbf{a}_k) = \operatorname{vol}_k(0\,\mathbf{0}, \mathbf{a}_2, \dots, \mathbf{a}_k) = 0 \operatorname{vol}_k(\mathbf{0}, \mathbf{a}_2, \dots, \mathbf{a}_k) = 0,$$

which proves (ii). QED

This brings us to the volume formula. We can form a matrix A out of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. It does not make sense to take $\det A$ because A is not square, unless $k = n$. However, the product $A^T A$ is square and we can take *its* determinant.

9.7. THEOREM. *There exists a unique k-dimensional volume function on $\mathbf{R}^n$. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathbf{R}^n$ and let A be the $n \times k$-matrix whose ith column is $\mathbf{a}_i$. Then*

$$\boxed{\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = \sqrt{\det(A^T A)}.}$$

PROOF. We leave it to the reader to check that the function $\sqrt{\det(A^T A)}$ satisfies the axioms for a k-dimensional volume function on $\mathbf{R}^n$. (See Exercise 9.3.) Here we prove only the uniqueness part of the theorem.

Case 1. First assume that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are orthogonal. Then $A^T A$ is a diagonal matrix. Its ith diagonal entry is $\|\mathbf{a}_i\|^2$, so $\sqrt{\det(A^T A)} = \|\mathbf{a}_1\|\,\|\mathbf{a}_2\| \cdots \|\mathbf{a}_k\|$, which is equal to $\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k)$ by Lemma 9.6(i).

Case 2. Next assume that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are dependent. Then the matrix A has a nontrivial nullspace, i.e. there exists a nonzero k-vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$. But then $A^T A\mathbf{v} = \mathbf{0}$, so the columns of $A^T A$ are dependent as well. Since $A^T A$ is square, this implies $\det A^T A = 0$, so $\sqrt{\det(A^T A)} = 0$, which is equal to $\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k)$ by Lemma 9.6(ii).

Case 3. Finally consider an arbitrary sequence of independent vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. This sequence can be transformed into an orthogonal sequence $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ by the Gram-Schmidt procedure. This works as follows: let $\mathbf{b}_1 = \mathbf{0}$ and for $i > 1$ let $\mathbf{b}_i$ be the orthogonal projection of $\mathbf{a}_i$ onto the span of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{i-1}$;

then $\mathbf{v}_i = \mathbf{a}_i - \mathbf{b}_i$. Let V be the $n \times k$-matrix whose ith column is $\mathbf{v}_i$. Then by repeated applications of Axiom (ii),

$$\begin{aligned}\text{(9.1)} \quad \operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) &= \operatorname{vol}_k(\mathbf{v}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = \operatorname{vol}_k(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{a}_k) = \cdots \\ &= \operatorname{vol}_k(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \sqrt{\det(V^T V)},\end{aligned}$$

where the last equality follows from Case 1. Since $\mathbf{v}_i = \mathbf{a}_i - \mathbf{b}_i$, where $\mathbf{b}_i$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{i-1}$, we have $V = AN$, where N is a $k \times k$-matrix of the form

$$N = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & & \ddots & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Note that N has determinant 1. This implies that $V^T V = N^T A^T A N$ and

$$\det(A^T A) = \det N^T \det(A^T A) \det N = \det(N^T A^T A N) = \det(V^T V).$$

Using (9.1) we get $\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = \sqrt{\det(A^T A)}$. QED

3. Volume forms

Now let M be an *oriented* n-manifold in $\mathbf{R}^N$. The *volume form* μ_M, or μ, is the n-form on M defined as follows. Let $\mathbf{x} \in M$ and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in T_{\mathbf{x}}M$. Then

$$\mu_{M,\mathbf{x}}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) \\ = \begin{cases} \operatorname{vol}_n(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) & \text{if } (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) \text{ is positively oriented,} \\ -\operatorname{vol}_n(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) & \text{if } (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) \text{ is negatively oriented,} \end{cases}$$

i.e. the oriented volume of the n-dimensional parallelepiped in $T_{\mathbf{x}}M$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

For $n = 1$ this is usually called the *element of arclength*, for $n = 2$, the *element of surface area*, and for $n = 3$, the *volume element*. Traditionally these are denoted by ds, dA, and dV, respectively. Don't be misled by this old-fashioned notation: volume forms are far from being exact!

The volume form μ_M is highly dependent on the embedding of M into $\mathbf{R}^N$. It changes if we dilate or shrink or otherwise deform M.

9.8. EXAMPLE. Let I be an interval in the real line and $f\colon I \to \mathbf{R}$ a smooth function. Let $M \subseteq \mathbf{R}^2$ be the graph of f. By Theorem 3.3 M is a 1-manifold in $\mathbf{R}^2$. Indeed, M is the image of the embedding $F\colon I \to \mathbf{R}^2$ given by $F(t) = (t, f(t))$. Let us give M the orientation induced by the embedding F, i.e. "from left to right". What is μ, the element of arclength of M? Let us compute the pullback $F^*\mu_M$, a 1-form on I. We have $F^*\mu_M = g(t)dt$ for a suitable function g on I, namely $g(t) = (F^*\mu_M)_t(\mathbf{e}_1) = \mu_{F(t)}(dF_t(\mathbf{e}_1))$. Here $dF_t = \binom{1}{f'(t)}$ and $\mathbf{e}_1$ is the standard basis vector on $\mathbf{R}^1 = \mathbf{R}$, i.e. the number 1. Consequently

$$\begin{aligned} g(t) &= \mu_{F(t)}\left(\begin{smallmatrix} 1 \\ f'(t) \end{smallmatrix}\right) \\ &= \text{length of } \left(\begin{smallmatrix} 1 \\ f'(t) \end{smallmatrix}\right) \\ &= \sqrt{1 + f'(t)^2}. \end{aligned}$$

Therefore $F^*\mu = \sqrt{1 + f'(t)^2}\,dt$.

9.9. EXAMPLE. More generally, let U be an open subset of $\mathbf{R}^n$ and $\psi\colon U \to \mathbf{R}^N$ an embedding. Then $M = \psi(U)$ is an n-manifold in $\mathbf{R}^N$ and the tangent space $T_{\psi(\mathbf{x})}M$ is $d\psi_{\mathbf{x}}(\mathbf{R}^n)$, the column space of $d\psi_{\mathbf{x}}$. We equip M with an orientation by demanding that $d\psi_{\mathbf{x}}\colon \mathbf{R}^n \to T_{\psi(\mathbf{x})}M$ be orientation-preserving for each $\mathbf{x} \in U$. Let μ be the volume form on M and let us calculate the n-form $\psi^*\mu$ on U. We have $\psi^*\mu = g\,dt_1 \cdots dt_n$ with

$$\begin{aligned} g &= \psi^*\mu(\mathbf{e}_1, \dots, \mathbf{e}_n) \\ &= \mu(d\psi(\mathbf{e}_1), \dots, d\psi(\mathbf{e}_n)) \\ &= \mu(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)) && \text{where } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \text{ are the column vectors of } d\psi \\ &= \sqrt{\det(d\psi^T d\psi)} && \text{by Theorem 9.7,} \end{aligned}$$

and therefore $\psi^*\mu = \sqrt{\det(d\psi^T d\psi)}\,dt_1 \cdots dt_n$.

Volume form of a hypersurface. For oriented hypersurfaces M in $\mathbf{R}^n$ there is a more convenient expression for the volume form μ than the one given in Example 9.9. Let $\mathbf{n}$ be the positive unit normal vector field on M and let α be a 1-form on $\mathbf{R}^{n+1}$. Then $*\alpha$ is an $n-1$-form, so on M we can write $*\alpha = f\mu$ for a certain function f defined on M. We can determine this function by substituting the vector field $\mathbf{n}$ into α.

9.10. LEMMA. *On the hypersurface M we have $*\alpha = \alpha(\mathbf{n})\mu$.*

PROOF. Let $\mathbf{x} \in M$. Since $T_{\mathbf{x}}M$ is an $n-1$-dimensional subspace of $\mathbf{R}^n$, we can choose a positively oriented orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ of $\mathbf{R}^n$ such that $(\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_{n-1})$ is a positively oriented basis of $T_{\mathbf{x}}M$. Then at $\mathbf{x}$ the positive unit normal is given by $\mathbf{n}(\mathbf{x}) = (-1)^{n+1}\mathbf{e}_n$ and the volume element satisfies $\mu_{\mathbf{x}}(\mathbf{e}_1, \dots, \mathbf{e}_{n-1}) = 1$. Writing $\alpha = \sum_{i=1}^n f_i\,dx_i$, we have $\alpha(\mathbf{n}(\mathbf{x})) = (-1)^{n+1} f_n(\mathbf{x})$. On the other hand

$$*\alpha = \sum_i (-1)^{i+1} f_i\,dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1},$$

and therefore $(*\alpha)(\mathbf{e}_1, \dots, \mathbf{e}_{n-1}) = (-1)^{n+1} f_n$. This proves that

$$(*\alpha)_{\mathbf{x}}(\mathbf{e}_1, \dots, \mathbf{e}_{n-1}) = \alpha(\mathbf{n}(\mathbf{x}))\mu(\mathbf{e}_1, \dots, \mathbf{e}_{n-1}),$$

which implies $(*\alpha)_{\mathbf{x}} = \alpha(\mathbf{n}(\mathbf{x}))\mu$. Since this holds for every $\mathbf{x} \in M$, we have $*\alpha = \alpha(\mathbf{n})\mu$. QED

It is convenient to rewrite this in terms of the the symbolic notations

$$d\mathbf{x} = (dx_1, dx_2, \dots, dx_n) \qquad \text{and} \qquad *d\mathbf{x} = (*dx_1, *dx_2, \dots, *dx_n).$$

If $F = (F_1, \dots, F_n)$ is a vector field on $\mathbf{R}^n$ we write $F \cdot d\mathbf{x}$ for the 1-form $\sum_i F_i\,dx_i$ and $F \cdot (*d\mathbf{x})$ for the n-form $\sum_i F_i(*dx_i)$. Observe that $\alpha(\mathbf{n})$ can be interpreted as the inner product $F \cdot \mathbf{n}$. Therefore Lemma (9.10) can be stated as

$$\boxed{F \cdot (*d\mathbf{x}) = (F \cdot \mathbf{n})\mu.}$$

In particular we can take $F = \mathbf{n}$, in which case $F \cdot \mathbf{n} = 1$. The upshot is the following expression for the volume form.

9.11. COROLLARY. *Let* $\mathbf{n}$ *be the unit normal vector field and* μ *the volume form of an oriented hypersurface in* $\mathbf{R}^n$. *Then*

$$\boxed{\mu = \mathbf{n} \cdot (*d\mathbf{x}).}$$

9.12. EXAMPLE. Suppose the hypersurface M is given by an equation $f(\mathbf{x}) = c$, where c is a regular value of a function $f\colon U \to \mathbf{R}$, with U open in $\mathbf{R}^n$. Then by Proposition 9.3 M has a unit normal $\mathbf{n} = \operatorname{grad} f / \|\operatorname{grad} f\|$. The volume form is therefore $\mu = \|\operatorname{grad} f\|^{-1} \operatorname{grad} f \cdot (*d\mathbf{x})$.

Exercises

9.1. Show that the unit normal vector field $\mathbf{n}\colon M \to S^{n-1}$ defined in the proof of Proposition 9.2 is smooth. (Compute $\mathbf{n}$ in terms of an orientation-preserving parametrization $\psi\colon U \to M$ of an open subset of M.)

9.2. Deduce from Theorem 9.7 that the area of the parallelogram spanned by a pair of vectors $\mathbf{a}_1$, $\mathbf{a}_2$ in $\mathbf{R}^3$ is given by $\|\mathbf{a}_1\|\,\|\mathbf{a}_2\|\,|\sin\phi|$, where ϕ is the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$.

9.3. Check that the function $\operatorname{vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = \sqrt{\det(A^T A)}$ satisfies the axioms of Definition 9.5.

9.4. Let $f\colon (a, b) \to \mathbf{R}^n$ be an embedding (as defined in Section 3). Let μ be the element of arclength on the embedded path $M = f(a, b)$. Show that $f^*\mu$ is the 1-form on (a, b) given by $\|f'(t)\|\,dt = \sqrt{f_1'(t)^2 + f_2'(t)^2 + \cdots + f_n'(t)^2}\,dt$.

9.5. (i) Let U be an open subset of $\mathbf{R}^n$ and let $f\colon U \to \mathbf{R}$ be a function satisfying $\operatorname{grad} f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x}$ in U. On U define a vector field $\mathbf{n}$, an $n-1$-form ν and a 1-form α by

$$\begin{aligned} n(\mathbf{x}) &= \|\operatorname{grad} f(\mathbf{x})\|^{-1} \operatorname{grad} f(\mathbf{x}), \\ \nu &= \mathbf{n} \cdot (*d\mathbf{x}), \\ \alpha &= \|\operatorname{grad} f(\mathbf{x})\|^{-1}\,df. \end{aligned}$$

Prove that $dx_1\,dx_2 \cdots dx_n = \alpha\nu$ on U.

(ii) Let $r\colon \mathbf{R}^n \to \mathbf{R}$ be the function $r(\mathbf{x}) = \|\mathbf{x}\|$ (distance to the origin). Deduce from (i) that $dx_1\,dx_2 \cdots dx_n = dr\,\nu$ on $\mathbf{R}^n - \{\mathbf{0}\}$, where $\nu = \|\mathbf{x}\|^{-1}\mathbf{x} \cdot (*d\mathbf{x})$.

9.6. Let V be the vector space of $n \times n$-matrices and let $f\colon V \to \mathbf{R}$ be the determinant function, $f(A) = \det A$.

(i) Show that

$$df(A)B = \sum_{i=1}^{n} \det(\mathbf{a}_1, \mathbf{a}_2 \dots, \mathbf{a}_{i-1}, \mathbf{b}_i, \mathbf{a}_{i+1}, \dots, \mathbf{a}_n),$$

where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ denote the column vectors of A, resp. B. (Use formula (3.1) and the multilinearity of the determinant.)

(ii) The *special linear group* is the subset of V defined by $\mathbf{SL}(n) = \{A : \det A = 1\}$. Show that $\mathbf{SL}(n)$ is a manifold. What is its dimension?

(iii) Show that for $A = I$, the identity matrix, we have $df(A)B = \sum_{i=1}^{n} b_{ii} = \operatorname{tr} B$, the *trace* of B. Conclude that the tangent space to $\mathbf{SL}(n)$ at I is the set of *traceless* matrices, i.e. matrices A satisfying $\operatorname{tr} A = 0$.