CHAPTER 10

Integration and Stokes' theorem on manifolds

The theory of chains and integration of forms over chains developed in Chapter 7 for open subsets of $\mathbf{R}^n$ carries over without difficulty to manifolds. The fact that such an integral is invariant under reparametrizations of the chain leads to the notion of integration of an n-form over an oriented n-manifold. In particular, by integrating the volume form we find the volume of the manifold.

1. Chains and Stokes for manifolds

Let M be an n-manifold and let R be a closed and bounded subset of $\mathbf{R}^k$. Let us call a map $c\colon R \to M$ *smooth* if it is the restriction of a smooth map $U \to M$ defined on an open subset U of $\mathbf{R}^k$ containing R. If α is a k-form on M we can pull α back along a smooth $c\colon R \to M$ to obtain a k-form on R (and on the larger open set U where c is defined). We can write $c^*\alpha = g(\mathbf{t})\,dt_1\,dt_2 \cdots dt_k$ for a smooth function and we define

$$\int_c \alpha = \int_R c^*\alpha = \int_R g(\mathbf{t})\,dt_1\,dt_2 \cdots dt_k.$$

The assumption that R be closed and bounded is made to ensure that the right-hand side is a proper multiple integral and therefore always exists.

A smooth map $c\colon [0,1]^k \to M$ is a k-*cube* in M and a formal linear combination of k-cubes with real coefficients, $\sum_{i=1}^p a_i c_i$ is a *(cubical)* k-*chain*. As in Chapter 7 we can define the boundary ∂c of k-chain c, which is a $k-1$-chain. The exact same arguments as in that chapter give $\partial^2 = 0$ and Stokes' theorem.

10.1. THEOREM (Stokes' theorem for manifolds; version I). *Let α be a $k-1$-form on a manifold M and let c be a k-chain in M. Then*

$$\boxed{\int_c d\alpha = \int_{\partial c} \alpha.}$$

2. Integration over orientable manifolds

As in Chapter 7 the change of variables formula for integrals gives the following result.

10.2. THEOREM. *Let α be a k-form on a manifold M and let $c\colon R \to U$ a smooth map, where R is closed and bounded in $\mathbf{R}^k$. Let $p\colon \bar{R} \to R$ be a reparametrization. Then*

$$\int_{c\circ p} \alpha = \begin{cases} \int_c \alpha & \textit{if } p \textit{ preserves the orientation} \\ -\int_c \alpha & \textit{if } p \textit{ reverses the orientation.} \end{cases}$$

Intuitively this says that to a large extent the integral depends only on α and on the set $c(R)$, not on the way the map c "traces out" the set $c(R)$. This opens up the possibility of integrating an n-form α over an n-manifold M: first we subdivide M into regions which look like n-dimensional cubes (in the sense that each is the image of an embedding $[0,1]^n \to M$), and which overlap only in their boundaries. Then we integrate α over each such subregion and add all contributions. Because the interiors of the subregions do not overlap there is no double counting, and Theorem 10.2 ought to guarantee that the answer is independent of the way in which we subdivided M!

This can indeed be done, but three difficulties need to be overcome. First, the integral is well-defined only if all the subregions can be oriented in a "coherent" fashion. Otherwise we can't be sure of the sign of the integral over each subregion. This obstacle is cleared simply by requiring M to be orientable and by fixing an orientation. (If we reverse the orientation of M the integral changes sign globally.) Second, the number of subregions may be infinite and the sum of the integrals may not converge. This danger is obviated by requiring M to be *compact*, i.e. a closed and bounded subset of $\mathbf{R}^N$. The third problem is trickier: how do we subdivide a manifold into cubical regions whose interiors do not overlap? This turns out to be possible but rather hard. There are easier ways of defining the integral.

We shall show how to do this only in a simple case and then state the general result. Let M be a compact oriented n-dimensional manifold and let α be an n-form on M. Suppose $c\colon [0,1]^n \to M$ is a smooth map and suppose that the restriction of c to the open n-cube $(0,1)^n$ is an orientation-preserving embedding. Suppose that $\alpha_{\mathbf{x}} = 0$ for all $\mathbf{x}$ in M outside the image $c([0,1]^n)$ of c. (In this case we say that α is *supported* on the image of c.) Then we define the *(oriented) integral* of α over M to be

$$\int_M \alpha = \int_c \alpha. \tag{10.1}$$

If $\bar{c}\colon [0,1]^n \to M$ is a second n-cube such that $\bar{c}\colon (0,1)^n \to M$ is an orientation-preserving embedding and $\alpha_{\mathbf{x}} = 0$ for $\mathbf{x}$ outside $c([0,1]^n)$, then Theorem 10.2 implies that $\int_c \alpha = \int_{\bar{c}} \alpha$. Thus the integral of α is well-defined.

It turns out that an arbitrary n-form α on M can be written as a sum of n-forms, each of which is supported on the image of a suitable n-cube. On the basis of this fact the integral $\int_M \alpha$ can then be characterized as follows.

10.3. THEOREM. *Let M be a compact oriented n-manifold. There is a unique function $\int_M \colon \Omega^n(M) \to \mathbf{R}$ satisfying*

(i) $\int_M (a_1\alpha_1 + a_2\alpha_2) = a_1 \int_M \alpha_1 + a_2 \int_M \alpha_2$ *for all scalars a_1 and a_2 and for all n-forms α_1 and α_2 on M;*

(ii) (10.1) *holds for any α supported on the image of an n-cube c in M such that the restriction of c to $(0,1)^n$ is an orientation-preserving embedding.*

The *volume* of a compact oriented manifold M in $\mathbf{R}^N$ is defined by $\operatorname{vol} M = \int_M \mu$, where μ is the volume form on M. More generally, the integral of a function f on M is defined as the integral of the n-form $f\mu$. The *mean* or *average* of f is the number $\bar{f} = (\operatorname{vol} M)^{-1} \int_M f\mu$. The *centroid* of M is the point $\bar{\mathbf{x}}$ in R^n whose coordinates are given by

$$\bar{x}_i = \frac{1}{\operatorname{vol} M} \int_M x_i \mu,$$

i.e. the mean value of x_i over M. Note carefully that the volume form depends on the particular embedding of M into $\mathbf{R}^N$, and so do the volume, the integral of a function, etc.

3. Manifolds with boundary and Stokes' theorem

Manifolds with boundary. Sailors of old imagined the surface of the earth to be not a sphere but a disc, and were afraid they might sail over the edge. A (closed) disc is not a manifold, because no neighbourhood of a point on the edge is diffeomorphic to an open subset of $\mathbf{R}^2$. It is an example of the more general notion of a manifold with boundary. The n-*dimensional halfspace* is

$$\mathbf{H}^n = \{ x \in \mathbf{R}^n : x_n \geq 0 \}.$$

The *boundary* of $\mathbf{H}^n$ is $\partial \mathbf{H}^n = \{ x \in \mathbf{R}^n : x_n = 0 \} = \mathbf{R}^{n-1}$ and its *interior* is $\operatorname{int} \mathbf{H}^n = \{ x \in \mathbf{R}^n : x_n > 0 \}$. n-dimensional manifolds with boundary are defined by replacing $\mathbf{R}^n$ with $\mathbf{H}^n$ in the definition of a n-dimensional manifold. More precisely, a n-*dimensional manifold with boundary* in $\mathbf{R}^N$ is a subset M such that for each $\mathbf{x} \in M$ there exist an open neighbourhood U of $\mathbf{x}$ in $\mathbf{R}^N$ and a smooth map $\phi \colon U \to \mathbf{R}^N$ satisfying

- ϕ is a diffeomorphism onto an open subset of $\mathbf{R}^N$;
- $M \cap U = \phi^{-1}(\mathbf{H}^n)$.

If $\phi(\mathbf{x})$ is in $\partial \mathbf{H}^n$, then $\mathbf{x}$ is a *boundary point* of M. The *boundary* of M is the set of all boundary points and is denoted by ∂M. Its complement is the *interior* of M and is denoted by $\operatorname{int} M$. If U and ϕ are as in the definition, then $\partial M \cap U = \phi^{-1}(\mathbf{R}^{n-1})$, so ∂M is an $n-1$-dimensional manifold. Likewise $\operatorname{int} M$ is an n-manifold.

Somewhat confusingly, the boundary of a manifold with boundary is allowed to be empty. If ∂M is empty, then M is a manifold in the old sense.

10.4. EXAMPLE. The halfspace $\mathbf{H}^n$ is an n-manifold with boundary $\partial \mathbf{H}^n = \mathbf{R}^{n-1}$ and interior $\mathbf{H} - \partial \mathbf{H}^n = \{ x \in \mathbf{R}^n : x_n > 0 \}$. A closed interval $[a, b]$ is a 1-manifold with boundary $\{a, b\}$. A closed ball $\bar{B}_\varepsilon(x) = \{ y \in \mathbf{R}^n : \|y - x\| \leq \varepsilon \}$ is a manifold with boundary the sphere of radius ε about x. The spherical shell

$$\{ x \in \mathbf{R}^n : R_1 \leq \|x\| \leq R_2 \}$$

is a manifold whose boundary is a union of two concentric spheres.

10.5. EXAMPLE. A square $[0, 1] \times [0, 1]$ is the product of two manifolds with boundary, but is itself not a manifold with boundary. The problem is caused by the sharp angles in the edge of the square, which are impossible to straighten out by a diffeomorphism.

10.6. EXAMPLE. A *Möbius band* is obtained by gluing together the two ends of a rectangular strip of paper after twisting one end by a half turn. (Make one at home!) The boundary results from two boundary edges AB and CD of the rectangle by gluing A to C and B to D. The boundary is therefore diffeomorphic to a circle S^1. The Möbius band is a nonorientable manifold with boundary. We will not give a proof of this fact, but you can convince yourself that it is true by making a Möbius band and trying to paint the two sides in different colours.

10.7. EXAMPLE. A *domain* is a manifold with boundary M in $\mathbf{R}^n$ such that $\operatorname{int} M$ is an open subset of $\mathbf{R}^n$. Note that M has the same dimension as the surrounding vector space $\mathbf{R}^n$. For instance, a closed ball is a domain in $\mathbf{R}^n$.

To define the *tangent space* to a manifold with boundary M at a point $\mathbf{x}$ choose U and ϕ as in the definition and put

$$T_\mathbf{x}M = (d\phi(\mathbf{x}))^{-1}(\mathbf{R}^n).$$

(*Not* $(d\phi(\mathbf{x}))^{-1}(\mathbf{H}^n)$!) As in the case of a manifold, this does not depend on the choice of the map ϕ. If $\mathbf{x}$ is a boundary point, then the tangent space at $\mathbf{x}$ to ∂M is

$$T_\mathbf{x}\partial M = (d\phi(\mathbf{x}))^{-1}(\mathbf{R}^{n-1}).$$

A tangent vector $\mathbf{v}$ at $\mathbf{x}$ with $d\phi(\mathbf{x})\mathbf{v} \in \operatorname{int}\mathbf{H}^n$ (i.e. the nth coordinate of $d\phi(\mathbf{x})\mathbf{v}$ is positive) is said to point *inwards*. If $d\phi(\mathbf{x})\mathbf{v} \in \mathbf{R}^n - \mathbf{H}^n$ (i.e. the nth coordinate of $d\phi(\mathbf{x})\mathbf{v}$ is negative) then $\mathbf{v}$ point *outwards*. Tangent vectors at $\mathbf{x}$ that point neither in- nor outwards are, of course, tangent to the boundary.

Orienting the boundary. Let M be any oriented manifold with boundary. The orientation on M induces an orientation on ∂M by a method very similar to the proof of Proposition 9.2. Namely, for $\mathbf{x} \in \partial M$ define $\mathbf{n}(\mathbf{x}) \in T_\mathbf{x}M$ to be the unique outward pointing tangent vector of length 1 which is orthogonal to $T_\mathbf{x}\partial M$. This defines the *unit outward-pointing normal vector field* on ∂M. A basis $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ of $T_\mathbf{x}\partial M$ is called positively oriented if $(\mathbf{n}(\mathbf{x}), \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1})$ is a positively oriented basis of $T_\mathbf{x}M$. This defines an orientation of ∂M, called the *induced orientation*. For instance, let $M = \mathbf{H}^n$ with the standard orientation $[\mathbf{e}_1, \dots, \mathbf{e}_n]$. At each point of $\partial M = \mathbf{R}^{n-1}$ the outward pointing normal is $-\mathbf{e}_n$. This implies that induced orientation on ∂M is $(-1)^n[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1}]$, because

$$[-\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1}] = (-1)^n[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1}, \mathbf{e}_n]$$

The most important property of the integral is the following version of Stokes' theorem, which is a parametrization-independent version of Theorem 10.1.

10.8. THEOREM (Stokes' theorem for manifolds; version II). *Let α be an $n-1$-form on a compact oriented manifold with boundary M. Give the boundary ∂M the induced orientation. Then*

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha.}$$

PROOF. In next year's version of the notes. QED

4. Gauß and Stokes

Stokes' theorem, Theorem 10.8, contains as special cases the integral theorems of vector calculus. These classical results involve a vector field $\mathbf{F} = \sum_{i=1}^n F_i\mathbf{e}_i$ defined on an open subset U of $\mathbf{R}^n$. To this vector field corresponds a 1-form $\alpha = \mathbf{F}\cdot d\mathbf{x} = \sum_{i=1}^n F_i\,dx_i$, which we can think of as the work done by the force $\mathbf{F}$ along an infinitesimal line segment $d\mathbf{x}$.

Fundamental theorem of calculus. If $\mathbf{F}$ is conservative, $\mathbf{F} = \operatorname{grad} g$ for a function g, then $\alpha = \operatorname{grad} g \cdot d\mathbf{x} = -dg$. If M is a compact oriented 1-manifold with boundary in $\mathbf{R}^n$, then $\int_M dg = \int_{\partial M} g$ by Theorem 10.8. The boundary consists of two points $\mathbf{a}$ and $\mathbf{b}$ (if M is connected). If the orientation of M is "from $\mathbf{a}$ to $\mathbf{b}$", then $\mathbf{a}$ acquires a minus and $\mathbf{b}$ a plus. Written in terms of $\mathbf{F}$ Stokes' theorem therefore says

$$\boxed{\int_M \mathbf{F} \cdot d\mathbf{x} = g(\mathbf{b}) - g(\mathbf{a}).}$$

If we interpret $\mathbf{F}$ as a force acting on a particle travelling along M, then $-g$ stands for the potential energy of the particle in the force field. Thus the potential energy of the particle decreases by the amount of work done.

Gauß' divergence theorem. We have

$$*\alpha = \mathbf{F} \cdot *d\mathbf{x} \qquad \text{and} \qquad d{*\alpha} = \operatorname{div} \mathbf{F}\, dx_1\, dx_2 \cdots dx_n.$$

If N is a oriented hypersurface in $\mathbf{R}^n$ with positive unit normal $\mathbf{n}$, then $*\alpha = (\mathbf{F} \cdot \mathbf{n})\mu_N$. In this situation it is best to think of $\mathbf{F}$ as the flow vector field of a fluid, where the direction of $\mathbf{F}(\mathbf{x})$ gives the direction of the flow at a point $\mathbf{x}$ and the magnitude $\|\mathbf{F}(\mathbf{x})\|$ gives the mass of the amount of fluid passing per unit time through a surface of unit area placed at $\mathbf{x}$ perpendicular to the vector $\mathbf{F}(\mathbf{x})$. Then $*\alpha$ describes the amount of fluid passing per unit time and per unit area through the hypersurface N. For this reason the $n-1$-form $*\alpha$ is also called the *flux* of $\mathbf{F}$, and its integral over N the *total flux* through N.

Applying Stokes' theorem to a compact domain M in $\mathbf{R}^n$ we get $\int_M d{*\alpha} = \int_{\partial M} \alpha$. Written in terms of $\mathbf{F}$ this is Gauß' divergence theorem,

$$\boxed{\int_M \operatorname{div} \mathbf{F}\, dx_1\, dx_2 \cdots dx_n = \int_{\partial M} (\mathbf{F} \cdot \mathbf{n})\mu_{\partial M}.}$$

Thus the total flux out of the hypersurface ∂M is the integral of $\operatorname{div} \mathbf{F}$ over M. If the fluid is incompressible (e.g. most liquids) then this formula leads to the interpretation of $\operatorname{div} \mathbf{F}$ (or equivalently $d{*\alpha}$) as a measure of the sources or sinks of the flow. Thus $\operatorname{div} \mathbf{F} = 0$ for an incompressible fluid without sources or sinks. If the fluid is a gas and if there are no sources or sinks then $\operatorname{div} \mathbf{F}(\mathbf{x}) > 0$ (resp. < 0) indicates that the gas is evaporating away from (resp. condensating at) $\mathbf{x}$.

Classical version of Stokes' theorem. We can also consider α and $d\alpha$ and write Stokes' theorem $\int_M d\alpha = \int_{\partial M} \alpha$ for a compact two-dimensional oriented surface with boundary M. The right-hand side represents the work of $\mathbf{F}$ done around the boundary curve(s) of M, which is not necessarily 0 if $\mathbf{F}$ is not conservative. The left-hand side has a nice interpretation if $n = 3$. Then $*d\alpha = \operatorname{curl} \mathbf{F} \cdot d\mathbf{x}$, so $d\alpha = \operatorname{curl} \mathbf{F} \cdot *d\mathbf{x}$. Hence if $\mathbf{n}$ is the positive unit normal of the surface M in $\mathbf{R}^3$, then $d\alpha = (\operatorname{curl} \mathbf{F} \cdot \mathbf{n})\mu_M$ on M. In this way we get the classical formula of Stokes,

$$\boxed{\int_M (\operatorname{curl} \mathbf{F} \cdot \mathbf{n})\mu_M = \int_{\partial M} \mathbf{F} \cdot d\mathbf{x}.}$$

In other words, the total flux of $\operatorname{curl} \mathbf{F}$ through the surface M is equal to the work done by $\mathbf{F}$ around the boundary curves of M.

Exercises

10.1. (i) Let $\alpha = x\,dy - y\,dx$ and let M be a domain in the plane $\mathbf{R}^2$. Show that $\int_{\partial M} \alpha$ is twice the surface area of M.

(ii) Apply the observation of (i) to find the area enclosed by the astroid $x = \cos^3 t$, $y = \sin^3 t$.

(iii) Let $\alpha = \sum_{i=1}^n (-1)^{i+1} x_i\,dx_1 \cdots \widehat{dx_i} \cdots dx_n$ and let M be a domain in $\mathbf{R}^n$. Show that $\int_{\partial M} \alpha$ is a constant times the volume of M. What is the value of the constant?

10.2 (ΕΥΡΗΚΑ). Write the divergence theorem for the vector field $\mathbf{F} = -cx_n\mathbf{e}_n$ on $\mathbf{R}^n$. Deduce Archimedes' Law: the buoyant force exerted on a submerged body is equal to the weight of the displaced fluid.

10.3. Let M be a compact domain in $\mathbf{R}^n$. Let f and g be smooth functions on M. The *Dirichlet integral* of f and g is $D(f, g) = \int_M (\operatorname{grad} f \cdot \operatorname{grad} g)\mu$, where $\mu = dx_1 \cdots dx_n$ is the volume form on M.

(i) Show that $df(*dg) = (\operatorname{grad} f \cdot \operatorname{grad} g)\mu$.

(ii) Show that $d(*dg) = (\Delta g)\mu$, where $\Delta g = \sum_{i=1}^n \partial^2 g/\partial x_i^2$.

(iii) Deduce from (i)–(ii) that $d(f(*dg)) = (\operatorname{grad} f \cdot \operatorname{grad} g + f\Delta g)\mu$.

(iv) Deduce from (iii) Green's formula, $\displaystyle\int_{\partial M} f(*dg) = D(f, g) + \int_M (f\Delta g)\mu$.

(v) And now Green's symmetric formula,

$$\int_{\partial M} \big(f(*dg) - g(*df)\big) = \int_M (f\Delta g - g\Delta f)\mu.$$

(vi) Let $\mathbf{n}$ be the outward unit normal vector field on ∂M. Write $\partial f/\partial \mathbf{n}$ for the directional derivative $(df)\mathbf{n} = \operatorname{grad} f \cdot \mathbf{n}$. Show that

$$\int_{\partial M} \Big(f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\Big)\mu_{\partial M} = \int_M (f\Delta g - g\Delta f)\mu.$$

10.4. Let B be the ball of radius R about the origin in $\mathbf{R}^n$. Its boundary ∂B is the sphere of radius R. Put $V_n(R) = \operatorname{vol}_n B$ and $A_n(R) = \operatorname{vol}_{n-1} \partial B$. Also put $V_n = V_n(1)$ and $A_n = A_n(1)$.

(i) Deduce from Corollary 9.11 that the restriction of ν to ∂B is the volume form on ∂B, where ν is as in Exercise 9.5(ii). Conclude that $A_n(R) = \int_{\partial B} \nu$.

(ii) Show that $V_n(R) = R^n V_n$ and $A_n(R) = R^{n-1} A_n$.

(iii) Let $f\colon [0, \infty) \to \mathbf{R}$ be a continuous function. Define $g\colon \mathbf{R}^n \to \mathbf{R}$ by $g(\mathbf{x}) = f(\|\mathbf{x}\|)$. Prove that

$$\int_B g\,\mu_B = \int_0^R f(r)A_n(r)\,dr = A_n \int_0^R f(r)r^{n-1}\,dr.$$

(iv) Show that

$$\left(\int_{-\infty}^{\infty} e^{-r^2}\,dr\right)^n = A_n \int_0^{\infty} e^{-r^2} r^{n-1}\,dr.$$

(Take $f(r) = e^{-r^2}$ in (iii) and let $R \to \infty$.)

(v) Using Problems 2.6 and 2.7 conclude that

$$A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \quad \text{whence} \quad A_{2m} = \frac{2\pi^m}{(m-1)!} \quad \text{and} \quad A_{2m+1} = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m-1)}.$$

(vi) By taking $f(r) = 1$ in (iii) show that

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}, \quad \text{whence} \quad V_{2m} = \frac{\pi^m}{m!} \quad \text{and} \quad V_{2m+1} = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}.$$

(vii) Prove that $A_n(R) = \partial V_n(R)/\partial R$.

(viii) Complete the following table. (Conventions: a space of negative dimension is empty; the volume of a 0-dimensional manifold is its number of points.)

n	0	1	2	3	4	5
$V_n(R)$			πR^2	$\frac{4}{3}\pi R^3$		
$A_n(R)$			$2\pi R$			

(ix) Find $\lim_{n\to\infty} V_n$ and $\lim_{n\to\infty} A_n$. Use Stirling's formula,

$$\lim_{x\to\infty} \frac{\Gamma(x+1)e^x}{x^{x+\frac{1}{2}}} = \sqrt{2\pi}.$$