Math 418

Prelim, March 29,2001

Solutions

1. (20 points) Classify the singular points on the extended plane for the function $f(z) = (\sin z)^{-1}$. Evaluate the residue at every isolated singular point. Compute $\int_C f(z)\ dz$ where $C = \{|z-5| = 4\}$.

Solution. Since $\sin z$ is an entire function, f is analytic except, maybe at zeroes of $\sin z$. The equation $\sin z = 0$ is equivalent to $e^{iz} - e^{-iz} = 0$ or to $E^{2iz} = 1$. The solutions are $z = k\pi$ with integral k. Since $\cos k\pi \neq 0$, these are simple zeros for $\sin z$ and therefore they are simple poles for f. Point ∞ is not an isolated singularity.

$Res_f(k\pi) = 1/(\cos k\pi) = (-1)^k$. Two isolated singularities π and 2π are inside C. The sum of the corresponding residues is equal to 0.

2. (15 points) Find the Laurent series and the residue at point 0 for $f(z) = z^2 \sin \frac{1}{z}$.

Solution. Since $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$, we have

$$f(z) = z - \frac{1}{6z} + \dots$$

and $a_{-1} = -\frac{1}{6}$.

3. (25 points) The function $\Gamma(z)$ is analytic at 1 and satisfies the conditions $\Gamma(1) = 1, \Gamma(z+1) = z\Gamma(z)$ for every z. Which of the integers $\dots, -2, -1, 0, 1, 2, \dots$ are singularities for Γ? Describe the type of each singularity.

Solution. By induction,

$$\Gamma(z) = (z-1)(z-2)\dots(z-n+1)\Gamma(z-n+1)$$

for $n \geq 1$ and therefore Γ is analytic at $n = 1, 2, 3, \dots$

Also, by induction, for $n \geq 1$, $\Gamma(z) = h_n(z)/(z+n-1)$ where

$$h_n(z) = \frac{\Gamma(z+n)}{z(z+1)\dots(z+n-2)}.$$

Note that h_n is analytic at $-n+1$ and $h_n(-n+1) = \Gamma(1)/(n+1)(-n+1)(-n+2)\dots(-1) \neq 0$. Hence, $-n+1$ is a simple pole for $n = 1, 2, 3, \dots$.

Remark. A function which satisfies all conditions in Problem 3 can be constructed by the formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\ dt.$$

This is the celebrated Euler's Gamma function.

4. (10 points) Find the winding number for $f(z) = z^2(1-2z)^{-1}$ on the counterclockwise oriented circle $\{|z| = 1\}$.

Solution. $w_C = N - P = 2 - 1 = 1$.

5. Which of the following statements is true and which is false. Justify your answers.

(a) (7 points) If an entire function f is not a constant, then $|f(z)| < 10^{-10}$ at some point z.

Solution. True. Suppose that $|f(z)| \geq 10^{10}$ for all z. Then f has no zeros and $f(z)^{-1}$ is a bounded entire function. By the Liouville's theorem, $f(z) = \text{const.}$ which contradicts our assumption.

(b) (4 points) If α is a pole for f and for g, then it is a pole for $f + g$.

Solution. False. 0 is a pole for $f(z) = 1/z$ and for $g(z) = -1/z$ but it is not a pole for $f(z) + g(z) = 0$.

(c) (5 points) If α is a pole for f and an essential singularity for g, then it is an essential singularity for $f + g$.

Solution. True. In the principle part of f only a finite number of coefficients are not equal to 0. In the principle part of g infinite many coefficients are not equal to 0. Hence, the principle part of $f + g$ contains infinite many non-zero coefficients.

(d) (6 points) The polynomial $P(z) = z^4 - 7z - 1$ has four zeros in the disk $\{|z| < 2\}$.

Hint. Apply the Rouche's theorem to $f(z) = z^4$ and $g(z) = -7z - 1$.

Solution. True. On $C = \{|z| = 2\}$, $|f(z)| = 2^4 = 16 > 15 = 7|z| + 1 \geq |g(z)|$. By the Rouche's theorem, f and $P = f + g$ have the same number of zeros inside C and f has a zero of order 4 at 0.

(e) (8 points) The polynomial $P(z) = z^4 - 7z - 1$ has no zeros in the disk $\{|z| < 1\}$.

Hint. Apply the Rouche's theorem to $f(z) = -7z - 1$ and $g(z) = z^4$.

Solution. False. On $C = \{|z| = 1\}$, $|f(z)| = |-7z-1| \geq 7|z|-1 = 6 > 1 = |g(z)|$. By the Rouche's theorem, f and $P = f + g$ have the same number of zeros inside C and this number is equal to 1 for f.

6!!! Prove that if P and Q are polynomial of x and y with real coefficients and if $f(x + iy) = P(x, y) + iQ(x, y)$ is analytic at 0, then it is analytic at every $z \in \mathbb{C}$.

Solution. If a polynomial $f(x, y)$ of real variables x, y vanishes in a neighborhood of $(0, 0)$, then all its partials at $(0, 0)$ vanish and therefore all coefficients of f are equal to 0. We conclude that if the Cauchy-Riemann equations hold for P and Q in a neighborhood of 0, then they hold everywhere.