MATH 418 COMPLEX VARIABLES
Homework 1 Solution

Due January 30, 2001

C1. Solution: Expand the left side of this equality into $\sum_{k=0}^{k=n} \binom{n}{k} (\cos\theta)^k (i\sin\theta)^{(n-k)}$. Then compare the real parts and imaginary parts at both sides. $\square$

C2. Proof: Let z be $x+iy$ with x, y real numbers. Then, plug this representation into $w = \frac{z-1}{z+1}$, and ratioanlize the denominator, we get

$$w = \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2}$$

So, w is imaginary if and only if $x^2 + y^2 = 1$ holds. This is obviously equivalent to the condition that $|z| = 1$. Since w exactly stands for the angle determined by $-1, z, 1$, we are done. $\square$

C3. Proof: The polar coordinate representation of $1+i$ is $\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$. So, by the formula in Problem 1 with $\theta = \frac{\pi}{4}$, we get the desired equality.

Let's denote I as

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots$$

and II as

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{N}{7} + \ldots.$$

By the same trick applied in Problem 1, we expand $(1+i)^n$ by binomial formula and get $(1+i)^n = I + iII$. So, it's clear that I is $2^{\frac{n}{2}} \cos\frac{n\pi}{4}$ and II is $2^{\frac{n}{2}} \sin\frac{n\pi}{4}$. $\square$

C4. Proof: For the first two claims, use the polar coordinate representation of a complex number. For the rest of the claims, use the standard representation of a complex number, i.e. $z = x + iy$, with x, y real numbers. $\square$

C5. Solution: The basic idea here is to replace z with its polar coordinate representation or its standard representation, and then find some equations which have geometric meanings.

(i) $Rez > 0$ defines the right half plane.
(ii) $b_1 < \Im z < b_2$ defines a strip delineated by the straight lines $y = b_1$ and $y = b_2$.
(iii) $\Re\alpha z = a$ defines a straight line. (Recall the geometric meaning of $\Re\alpha z = a$.)
(iv) $|z - \alpha| = r$ defines a circle centered at α with radius r.
(v) $|z - \alpha| < r$ defines a disc without boundry and with α the center, r the radius.
(vi) $r_1 \leq |z| \leq r_2$ defines an anulus centered at 0.
(vii)$\Re\frac{1}{z} = 1$ defines the circle centered at $\frac{1}{2}$ with radius $\frac{1}{2}$. $\square$

C6. Solution:

$$(1+i)^2 = 2i, \frac{3+4i}{1-2i} = -1 + 2i$$

$$z^3 = (x^3 - 3xy^2) + (3yx^2 - y^3)i, \bar{z}z = x^2 + y^2$$

$$\frac{\bar{z}}{z} = \frac{x^2 - y^2 - 2xyi}{x^2 + y^2}, \frac{z - i}{1 - i\bar{z}} = \frac{2x - 2xy + i[x^2 - (y-1)^2]}{x^2 + (y-1)^2}.\square$$

C7. Proof: Use the polar coordinate representation of z: $z = r\exp\{i\alpha\}$, we have $\Re(1/z) > 0$ if and only if $\Re\frac{1}{r}\exp\{-i\alpha\} > 0$, i.e. $\cos\alpha > 0$ and $\sin\alpha = 0$. So, $\alpha = 2k\pi$, for some integer k. By similar reasoning, we can find out $\Re z > 0$ if and only if α satisfies the above condition. So, $\Re\frac{1}{z} > 0$ if and only if $\Re z > 0$. $\square$

C8. Solution: To see what transformation is going on, we will regard z as a point (r, α) in the plane, where $z = r\exp\{i\alpha\}$ is the polar coordinate representation of z.
(i)$z' = iz : (r, \alpha) \to (r, \alpha + \frac{\pi}{2})$. This is a counter clockwise rotation at the degree $\frac{\pi}{2}$.
(ii)$z' = 2z : (r, \alpha) \to (2r, \alpha)$. This is an expansion along the direction of z.
(iii)$z' = -z : (r, \alpha) \to (r, \alpha + \pi)$. This is a counter clockwise rotation at the degree π.
(iv)$z' = -2iz : (r, \alpha) \to (2r, \alpha - \frac{\pi}{2})$. So, this is a rotation combined with an expansion. $\square$

C9. Solution: Let $\alpha = 2 + 3i, \beta = 5 + 7i$. Then, by the fact $|\alpha\beta|^2 = |\alpha|^2|\beta|^2$, we have

$$962 = 13 \times 74 = |\alpha|^2 \times |\beta|^2 = |\alpha\beta|^2 = |-11 + 29i|^2 = 11^2 + 29^2$$

$\square$