MATH 418 COMPLEX VARIABLES
Homework 11 Solution

Due April 26, 2001

Note: If you have any questions about the solution, or you think there are some typos/errors in the solution, please e-mail me. I'll double-check it and then reply to you. Thank you.

C20. We are going to prove the hint first, then prove the result in the problem. Please note the lemmas below are very important. I spent a lot of time typing it out, in the hope you will benefit from it.

Lemma 1 (Schwarz Lemma): If $f(z)$ is an analytic mapping from open unit disc D onto D, and $f(0) = 0$, then $f|z| \leq |z|$ and $|f'(0)| \leq 1$. In case theequality $|f'(0)| = 1$ holds, $f(z)$ is a rotation, i.e. $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.
Proof: Define

$$\phi(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0 \\ f'(0), & \text{z=0} \end{cases}$$

By Morera's Theorem (page 137), $\phi(z)$ is analytic in D. $\forall z_0 \in D$, $\exists r$, such that $|z_0| < r < 1$. By Maximum Principle, $|\phi(z_0)| \leq max_{|z|=r}|\phi(z)| \leq 1/r$. Let $r \longrightarrow 1$, we get $|\phi(z_0)| \leq 1$. Since z_0 is arbitrarily chosen, $|\phi(z)| \leq 1$ for any z in D, i.e. $|f(z)| \leq |z|$. In particular, $|f'(0)| = |\phi(0)| \leq 1$. If $|f'(0)| = 1$, then $|\phi(0)| = 1$. By Maximum Principle, $|\phi(z)|$ is indentical to 1 in D. So, $\phi(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$, i.e. $f(z) = ze^{i\theta}$. □

Lemma 2: If f is a 1-1 analytic function from D onto D and $f(0) = 0$, then $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.
Proof: Let g=f^{-1}. Then g is also a 1-1 analytic function form D onto D, and $g(0) = 0$. By Schwarz Lemma, $|g'(0)| \leq 1$, $|f'(0)| \leq 1$. But $f \circ g(z) = z$, which means $f'(g(z))g'(z) = 1$ for any z in D. Let $z = 0$ and note $g(0) = 0$, we get $f'(0)g'(0) = 1$. So $|f'(0)| = |g'(0)| = 1$. Again by Schwarz Lemma, $f(z)$ is a rotation. □

Proof of the hint: Since f is onto D, $\exists \alpha \in D$, such that $f(\alpha) = 0$. Let $g(z) \triangleq \frac{z-\alpha}{z\bar{\alpha}-1}$. Then it's easy to check g is a 1-1 analytic mapping from D onto D. So is $h \triangleq f \circ g$. Furthermore, we have $h(0) = 0$. By Lemma 2, $h(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$, i.e. $f \circ g(z) = e^{i\theta}z$. Let $g(z) = \zeta$, we get $z = \frac{\zeta-\alpha}{\zeta\bar{\alpha}-1}$. So, $f(\zeta) = e^{i\theta}\frac{\zeta-\alpha}{\zeta\bar{\alpha}-1}$. □

Proof of the problem: Let $g(z) = \frac{z-z_0}{z\bar{z}_0-1}$. g is a 1-1 analytic mapping from D onto D. So is $h(z) = g^{-1} \circ f \circ g(z)$ and we have $h(0) = 0$. By Lemma 2, $h(z)$ is a rotation $e^{i\theta}z$. But

$$h'(0) = (g^{-1})'(f \circ g(0))f'(g(0))g'(0) = \frac{1}{g'(f \circ g(0))} \times f'(z_0) \times g'(0) = f'(z_0) > 0$$

So, $e^{i\theta} = 1$. Hence $h(z) = z$ and $f(g(z)) = g(z)$. Since $g(z)$ is from D onto D, we conclude $f(z) = z$, $\forall z \in D$. □

C21. Solution: Replace α, β in the hint of problem 1 with z_0 and $|z_0|^2 - 1$. □

1. (a) Proof:

$$s(z) = \begin{cases} 0, & \text{if } 0 \le z < 1 \\ 1, & \text{if } z = 0 \end{cases}$$

If the convergence is uniform, since $z^n \in C[0,1]$, $s(z)$ should also be continuous over $[0,1]$. Contradiction. $\square$

(b) $s(z) = 0$. Assume the convergence is uniform, then $\forall \epsilon \in (0,1)$, $\exists M$, such that $\forall n \ge M$, $z^n < \epsilon$. In particular, let $\epsilon = 1/2$, then for any $n \in \mathbb{N}$, we can find $z_n = \left(\frac{3}{4}\right)^{1/n} \in [0,1)$ and $z^n = 3/4 > 1/2$. Contradiction. $\square$

2. Note $|s_n(z)+t_n(z)-(s(z)+t(z))| \le |s_n(z)-s(z)|+|t_n(z)-t(z)|$, $|s_n(z)f(z)-s(z)f(z)| \le C|s_n(z)-s(z)|$, where $C = \sup_{z\in G}|f(z)|$. Then prove by checking definition of uniform convergence.

$s_n(z) \longrightarrow 0$ uniformly in punctured unit disc, $f(z) = 1/z$ is well-defined on punctured unit disc. But $s(z)f(z) = 0$, and $s_n(z)f(z) = 1/(zn)$ doesn't converges to 0 uniformly ($\forall n$, you just pick up a $z_n = 1/n$). $\square$

2. Solution: Use ratio test, the convergence radius R are, respectively, $1/2, 1, 1/2, 5/3$, and e. $\square$

5. Solution: The definition of meromorphic is in page 192, line 12 from the top.
(1) Single poles: $-n$, where $n = 0, 1, 2, \ldots$. Residue at $-n$ is $(-1)^n/n!$.
(2) Single poles: ki where $k \in \mathbb{Z} - \{0\}$. Residue at ki is $\frac{1}{2ki}$.
(3) Poles of order 2: $-n$ where $n \in \mathbb{N}$. Residue is 0.
(4) Single poles: ki where $k \in \mathbb{Z} - \{0\}$. Residue is $\frac{\sin n^2 i}{n!2ni}$. $\square$