MATH 418 COMPLEX VARIABLES
Homework 2 Solution

Due February 6, 2001

Note: If you have any questions about the solution, please e-mail me. I'll double-check it and then reply to you. Thanks.

C11. Proof:

$$f(z) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2} - i\frac{y(3x^2-y^2)}{x^2+y^2}, & \text{for } z \neq 0 \\ 0, & forz = 0 \end{cases}$$

So, $u(x,y) = \frac{x(x^2-3y^2)}{x^2+y^2}$ and $v(x,y) = -i\frac{y(3x^2-y^2)}{x^2+y^2}$. Then, it's easy to see

$$\frac{\partial u(0,0)}{\partial x} = 1 = \frac{\partial v(0,0)}{\partial y}, \frac{\partial u(0,0)}{\partial y} = 0 = -\frac{\partial v(0,0)}{\partial x}$$

So, $f(z)$ satisfies Cauchy-Riemann equations at the origin. But it's not differentiable at 0, because $\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \frac{\bar{z}^2}{z^2}$ does not exist. □

C12. Proof: $u(x,y) = x^3 + 3xy^2, v(x,y) = y^3 + 3x^2y$. So, u_x, u_y, v_x, v_y are continuous in the whole plane. According to the Theorem 1.1 in textbook, we only need to find the points where u, v satisfy Cauchy-Riemann equations.

$$u_x = 3x^2 + 3y^2, u_y = 6xy$$

$$v_x = 6xy, v_y = 3y^2 + 3x^2$$

So, u, v satisfy Cauchy-Riemann equations if and only $xy = 0$. This is to say, the points are on the coordinate axes. □

Problems in Levinson and Redheffer:

1. Solution:
(i)$f(z) = z + 1$, then $f(z+1) = z + 2, f(1/z) = 1/z + 1, f[f(z)] = z + 2$.
(ii)$f(z) = z^2$, then $f(z+1) = z^2 + 2z + 1, f(1/z) = 1/z^2, f[f(z)] = z^4$.
(iii)$f(z) = 1/z$, then $f(z+1) = 1/(z+1), f(1/z) = z, f[f(z)] = z$.
(iv)$f(z) = (z+1)(1-z)^{-1}$, then $f(z+1) = -(z+2)/z, f(1/z) = (z+1)(z-1)^{-1}, f[f(z)] = -1/z$. □

2. Solution: $Rez > 1, 0 < |z| < 1, Imz < 2|z|, |z-1| < |z+i|$ stand for domains. The last one, $2|z^2-1| < 1$, is not, sicne it's not connected (the existence of z^2 in the formula makes the pre-image consist of two sheaves). □

5. Solution: There are two ways of doing this. One is to take advantage of fractional linear transformation and do it in the setting of complex analysis. The other one is more or less traditional and elementary: suppose the transformation is T. First, regard z as a point (x, y) in the plane. Then $T(z)$ can be represented as $u(x,y) + iv(x,y)$, where u, v are real-valued functions. Finally, assume x, y can be represented by u and v, we replace x, y with u, v in the initial equation $F(x,y) = 0$ satisfied by x, y. Thus, we get equations of u and v. Another geometric method, which is more straightforward, is to decompose T into the composition of several simply transformations, and go step-by-step to check the resulted areas.

Define $|Rez| < 1$ as A, and $1 < Imz < 2$ as B.

(i)$T:(x,y)\longrightarrow(2x,2y+1)$. $T(A)$ is the strip between $x=2$ and $x=-2$. $T(B)$ is the strip between $y=3$ and $y=5$.

(ii)$T:(x,y)\longrightarrow(1+x-y,x+y)$. $T(A)$ is the strip between the line $x+y-3=0$ and the line $x+y+1=0$. $T(B)$ is the strip between $y-x-3=0$ and $y-x-1=0$.

(iii)$T:(x,y)\longrightarrow(2x^2-2y^2,4xy)$. $T(A)$ is the area defined by $y^2+8x-16<0$. $T(B)$ is the area defined by the $16+8x<y^2<64+16x$.

(iv)This is a fractional linear transformation. It's a homeomorphic from extendted complex plane to itself and tranforms circles to circles (lines are special cases of circles). So, $T(A)$ is the area surrounded by the circle passing $1,0,1/2-i/2$ (or, in terms of an equation of x and y, $(x-1/2)^2+y^2=1/4$),and the circle passing $-1,0,-1/2-i/2$ (or, $(x+1/2)^2+y^2=1/4$)–note it's the area out of these two circles, since $0\xrightarrow{1/z}\infty$. $T(B)$ is the area surrounded by the circle passing $-i,0,1/2-i/2$ (or, $(y+1/2)^2+x^2=1/4$) and the circle passing $0,-i/2,\frac{1-i}{4}$ (or, $(y+1/4)^2+x^2=1/16$).

(v)This is again a fractional linear transformation. So, $T(A)$ is the area surrouned by the line passing $1-2i,1$ (or, $x=1$) and the circle passing $0,(1-2i)/5,(1+2i)/5$ (or, $(x-1/2)^2+y^2=1/4$). $T(B)$ is the area surrounded by the circle passing $1,1-2i,-i$ (or,$(x-1)^2+(y+1)^2=1$) and the circle passing $1,1-i,\frac{3-4i}{5}$ (or, $(x-1)^2+(y+1/2)^2=1/4$).

□

2. Solution:
(a) Neither.
(b) The second one.
□

4. Proof: If $z\neq 0$, we then have $(z^{-1})'=\lim_{\triangle z\to 0}\frac{\frac{1}{z+\triangle z}-\frac{1}{z}}{\triangle z}=\lim_{\triangle z\to 0}-\frac{1}{(z+\triangle z)z}=-\frac{1}{z^2}$. According to chain rule, $(z^{-n})'=n(z^{-1})^{n-1}(z^{-1})'=-nz^{-n-1}$. □

10. Proof: The basic method is replacing$z$ with $x+iy$, and write the functions in the form of $u(x,y)+iv(x,y)$, where u,v are real-valued functions. Then, find the conditions about zeros and periods.

(1)$\cos z$: The period is 2π, the zeros are $k\pi+\pi/2(k\in\mathbb{Z})$.
(2)$\sin z$: The period is 2π, the zeros are $k\pi(k\in\mathbb{Z})$.
(3)$\cosh z$:The period is $i\pi$, the zeros are $k\pi+\pi/2(k\in\mathbb{Z})$.
(4)$\tan z$: The period is π, the zeros are $k\pi(k\in\mathbb{Z})$.
(5)$\tanh z$: The period is π, the zeros are $ik\pi(k\in\mathbb{Z})$.
□

14. Proof: The idea is to replace z with $x+iy$. And from the equations satisfied by z, we can get equations satisfied by x,y.
(1)For $e^z=i$, z should be $(2k\pi+\pi/2)i$, with $k\in\mathbb{Z}$.
(2)For $e^z=1+\sqrt{3}i=2e^{\frac{\pi}{3}i}$, z should be $\log 2+(2k\pi+\pi/3)i$, with $k\in\mathbb{Z}$.
(3)For $\sin z=2$, z should be $2k\pi+\pi/2+i\log(2\pm\sqrt{3})$, where $k\in\mathbb{Z}$.
(4)For $\cos^2 z=-1$, z should be $k\pi+\pi/2+i\log(\sqrt{2}\pm 1)$, where $k\in\mathbb{Z}$.
□