MATH 418 COMPLEX VARIABLES
Homework 3 Solution

Due February 13, 2001

Note: If you have any questions about the solution, or you think there are some typos/errors in the solution, please e-mail me. I'll double-check it and then reply to you. Thank you.

1a.
Solution:
(1) The cube root of 1 are 1, $e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$ and $e^{i4\pi/3} = -1/2 - i\sqrt{3}/2$.
(2) The cube root of -8 are -2, $2e^{i\pi/3} = 1 + i\sqrt{3}$, and $2e^{i5\pi/3} = 1 - i\sqrt{3}$.
(3) The cube root of i are $e^{i\pi/6} = \sqrt{3}/2 + i/2$, $e^{i5\pi/6} = -\sqrt{3}/2 + i/2$ and $-i$. $\square$

3.
Solution: In the following answers, the k stands for any integer.
(1) $\log(-i) = \log|-i| + iArg(-i) + 2k\pi i = \log 1 + i(-\pi/2) + 2k\pi i = (2k\pi - \pi/2)i$.
(2) $\log(1+i) = \log|1+i| + i\pi/4 + 2k\pi i = \log\sqrt{2} + (2k\pi + \pi/4)i$.
(3) $3\log(1+i\sqrt{3}) = 3\log 2 + 3i\pi/3 + 6k\pi i = 3\log 2 + i(6k+1)\pi$.
(4) $\log(1+i\sqrt{3})^3 = \log(-8) = \log|-8| + iArg(-8) + 2k\pi i = 3\log 2 + (2k\pi + 1)\pi$.
So, $\log(1+i\sqrt{3})^3 \neq 3\log(1+i\sqrt{3})$.
(5) $\log(1+i)^{i\pi} = i\pi\log(1+i) = i\pi(\log\sqrt{2} + i(\pi/4 + 2k\pi)) = -2k\pi^2 - \pi^2/4 + i(\pi\log\sqrt{2})$.
(6) We use (1) and get $(-i)^{-i} = e^{-i\log(-i)} = e^{-i(2k\pi-\pi/2)i} = e^{2k\pi-\pi/2}$.
(7) -1.
(8) 3^π.
(9) $e^{i\pi\log 2}$.
(10) We use the result in (2) and get $(1+i)^{1+i} = e^{(1+i)\log(1+i)} = e^{(1+i)[\log\sqrt{2}+(2k\pi+\pi/4)i])} = e^{\log\sqrt{2}-(2k\pi+\pi/4)+i[\log\sqrt{2}+2k\pi+\pi/4]} = e^{\log\sqrt{2}-(2k\pi+\pi/4)+i[\log\sqrt{2}+\pi/4]}$.
(11) 1. $\square$

2. We call the curves in the problem γ_1, γ_2, γ_3, γ_4,and γ_5, according to the order in the problem.
(b) Solution:
(i) $4z^3$ is analytic in the whole plane. So, if γ is a smooth curve starting at z_1, ending at z_2, then $\int_\gamma 4z^3dz = z_2^4 - z_1^4$. So,
$\int_{\gamma_1} 4z^3dz = (1+i)^4 - 1 = -5$.
$\int_{\gamma_2} 4z^3dz = (e^{-\pi i})^4 - 1 = 0$.
The third and fourth integral is done along a closed curve. So, the integrals are 0 in both cases.
$\int_{\gamma_5} 4z^3dz = (2+i)^4 - 1 = -8 + 24i$.
(ii)$\bar{z}$ is not analytic, we just replace $\bar{z}$ with its parametric representation and do the integration in the common way.
$\int_{\gamma_1} \bar{z}dz = \int_0^1 (1-it)idt = i + 1/2$.
$\int_{\gamma_2} \bar{z}dz = \int_0^1 e^{\pi ti}(-\pi i)e^{-\pi ti}dt = -\pi i$.
$\int_{\gamma_3} \bar{z}dz = \int_0^1 3e^{-2\pi ti}(6\pi ie^{2\pi ti})dt = 18\pi i$.
$\int_{\gamma_4} \bar{z}dz = \int_0^1 e^{-4\pi ti}(4\pi i)e^{4\pi i}dt = 4\pi i$.

$\int_{\gamma_5} \bar{z} dz = \int_0^1 (1 - it + t^2)(i + 2t) dt = \int_0^1 (2t^3 + 3t + (1 - t^2)i) dt = 2 + 2i/3.$
(iii) The most convincing method for this problem is replacing z with its parametric representation. While the most effective method is using log function. However, you should be careful when you are using this function since it may have multiply values at one point. Essentially speaking, when you calculate $\log z|_{z_2}^{z_1}$, you are appeal to its Riemann surface, whose rigorous definition is related to the covering space of the plane. All this may seem confusing, but for the time being, just remember one principle: if integral curve does not enclose origin point, or equivalently, there exists a domain which contains the integral curve but not the origin point, then $\log z_1$ and $\log z_2$ should have the same Arg.
$\int_{\gamma_1} dz/z = \log(1+i) - \log 1 = log(1+i) = \log \sqrt{2} + i\pi/4.$
$\int_{\gamma_2} dz/z = \int_0^1 \frac{-i\pi e^{-i\pi t}}{e^{-\pi t i}} dt = -i\pi.$
$\int_{\gamma_3} dz/z = \int_0^1 \frac{6i\pi e^{2\pi t i}}{3e^{2\pi t i}} dt = 2i\pi.$
$\int_{\gamma_4} dz/z = \int_0^1 \frac{4\pi i e^{4i\pi t}}{e^{4\pi t i}} dt = 4i\pi.$
$\int_{\gamma_5} dz/z = \log z|_1^{2+i} = \log 5 + i \arctan \frac{1}{2}.$

1.

Solution: The unit circle has the parametric representation $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$. So,
$\int_C \frac{dz}{z} = \int_0^{2\pi} i d\theta = 2\pi i.$
$\int_C \frac{dz}{|z|} = \int_C dz = 0.$
$\int_C \frac{|dz|}{z} = \int_0^{2\pi} e^{-i\theta} d\theta = 0.$
$\int_C \frac{dz}{z^2} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{2i\theta}} d\theta = \int_0^{2\pi} ie^{-i\theta} d\theta = -e^{-i\theta}|_0^{2\pi} = 0.$
$\int_C \frac{dz}{|z^2|} = \int_C dz = 0.$
$\int_C \frac{|dz|}{z^2} = \int_0^{2\pi} e^{-2i\theta} d\theta = 0. \square$

2.

Proof: $|\int_C \frac{dz}{4+3z}| \leq \int_C \frac{|dz|}{|4+3z|} \leq \int_C \frac{|dz|}{4-3|z|} = \int_C |dz| = 2\pi.$
Let us define C_1 as $\{|z| = 1, x < 0\}$, and C_2 as $\{|z| = 1, x \geq 0\}$. Then we have

$$|\int_C \frac{dz}{4+3z}| = |\int_{C_1} \frac{dz}{4+3z} + \int_{C_2} \frac{dz}{4+3z}| \leq \int_{C_1} \frac{|dz|}{4-3|z|} + \int_{C_2} \frac{|dz|}{|4+3x+i3y|}$$

$$= \pi + \int_{C_2} \frac{|dz|}{\sqrt{(4+3x)^2 + 9y^2}}$$

But $\sqrt{(4+3x)^2 + 9y^2} \geq \sqrt{16 + 24x + 9(x^2+y^2)} = \sqrt{25 + 24x} \geq 5$, since $z \in C_2$.
So, $\int_{C_2} \frac{|dz|}{\sqrt{(4+3x)^2+9y^2}} \leq \int_{C_2} \frac{|dz|}{5} = \pi/5$ and we get the desired inequality. $\square$