MATH 418 COMPLEX VARIABLES
Homework 4 Solution

Due February 20, 2001

Note: If you have any questions about the solution, or you think there are some typos/errors in the solution, please e-mail me. I'll double-check it and then reply to you. Thank you.

1.
Solution: We take the contour C as the circle $|z| = 2$. Replace z with its parametric representation $z = e^{i\theta}$, with $0 \leq \theta < 2\pi$. Then $\int_C f(z)dz = \int_0^{2\pi} e^{-i\theta} ie^{i\theta} d\theta = 2\pi i \neq 0$. So, the conclusion of Theorem 3.4 can fail if the region is not assumed to be a star domain.□

3.
Proof: WLOG, we assume $a > 0$. Take the contour C as the rectangle with vertices at $-b$, b, $b+ia$ and $-b+ia$, where b is a positive number. We take the orientation of C as clockwise. Let $f(z) = e^{-z^2}$, then $\int_C f(z)dz = \int_{-b}^{b} f(x+ia)dx + \int_a^0 f(b+iy)dy + \int_b^{-b} f(x)dx + \int_0^a f(-b+iy)dy$. We claim $\int_a^0 f(b+iy)dy$ and $\int_0^a f(-b+iy)dy$ go to 0, as $b \longrightarrow +\infty$. Indeed, we have

$$|\int_a^0 f(b+iy)dy| \leq \int_0^a |f(b+iy)|dy = \int_0^a |e^{-b^2}e^{-2byi}e^{y^2}|dy$$

$$= \int_0^a e^{-b^2}e^{y^2}dy \leq e^{-b^2}ae^{a^2} \overset{b\to+\infty}{\longrightarrow} 0$$

Similarly, $\int_0^a f(-b+iy)dy \longrightarrow 0$ as $b \longrightarrow +\infty$. Since $f(z)$ is analytic in the whole plane, $\int_C f(z)dz = 0$. Hence if we let $b \longrightarrow +\infty$, we have $0 = I(a) + \int_{\infty}^{-\infty} f(x)dx$. So, $I(a) = \int_{-\infty}^{\infty} f(x)dx$(constant independent of a).□

1.
Proof: Let $g(z) = e^{f(z)}$, then $g(z)$ is analytic is an entire function since it's the composition of two entire functions. Note $|g(z)| = |e^{Ref(z)+iImf(z)}| = e^{Ref(z)} \leq e^M$. So, $g(z)$ is bounded over the whole plane. By Liouville's theorem, $g(z)$, and hence $f(z)$, is constant. □

9a. Note: You can not take it for granted that you can take derivative inside the integration signal. You have to PROVE that.
Proof: $\forall h \in \mathbb{C}$, we only need to show $|\frac{F(z+h)-F(z)}{h} - i\int_a^b e^{izt}tf(t)dt| \longrightarrow 0$ as $h \longrightarrow 0$. Indeed, we have

$$|\frac{F(z+h)-F(z)}{h} - i\int_a^b e^{izt}tf(t)dt| = |\int_a^b [\frac{e^{iht}-1}{h} - it]e^{izt}f(t)dt| \leq \int_a^b |\frac{e^{iht}-1}{h} - it||e^{izt}f(t)|dt$$

Since $f(t)$ is continuous, it's bounded over the compact set $[a,b]$. Let $M = \max_{a\leq t\leq b}|f(t)|$, then

$$|\frac{F(z+h)-F(z)}{h} - i\int_a^b e^{izt}tf(t)dt| \leq M(b-a)\sup_{a\leq t\leq b}|\frac{e^{iht}-1}{h} - it|$$

Replace iht with ω. Since t is between a and b, $\omega \longrightarrow 0$ uniformly as $h \longrightarrow 0$. Note $|\frac{e^{iht}-1}{h} - it| = |\frac{e^{\omega}-1}{\omega}it - it| \leq |\frac{e^{\omega}-1}{\omega} - 1|(|a| \vee |b|)$. By the definition of derivative of an entire function at one point, in particularly, the derivative of e^{ω} at 0, we know $|\frac{e^{\omega}-1}{\omega}-1| \longrightarrow 0$ as h, and hence ω, goes to 0. Thus we achieved the aim we claimed at the beginning. $\square$