MATH 418 COMPLEX VARIABLES
Homework 5 Solution

Due February 27, 2001

Note: If you have any questions about the solution, or you think there are some typos/errors in the solution, please e-mail me. I'll double-check it and then reply to you. Thank you.

8.

Proof: $\forall z \notin C$, we only need to prove $F(z)$ is analytic in a neighbourhood of z. Indeed, for any given z, we define $d \triangleq \inf_{z' \in C} |z - z'|$. Intuitively, this is the distance between z and the contour C. Since $\{z\}$, C are compact and $z \notin C$, d is a positive number. Let $U(z, r)$ be the ball centered at z with radius $r < d$. $\forall z' \in U$, let $\Delta z = z' - z$, then

$$\left| \frac{F(z') - F(z)}{z' - z} - \int_C \frac{f(\zeta}{(\zeta - z)^2} d\zeta \right| = \left| \int_C \frac{f(\zeta)}{(\zeta - z')(\zeta - z)} d\zeta - \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right|$$

$$\leq \int_C |f(\zeta)| \left| \frac{\Delta z}{(\zeta - z')(\zeta - z)^2} \right| |d\zeta|$$

We estimate the kernel $|\frac{\Delta z}{(\zeta - z')(\zeta - z)^2}|$ as follows

$$\left| \frac{\Delta z}{(\zeta - z')(\zeta - z)^2} \right| = \frac{|\Delta z|}{|\zeta - z - \Delta z||\zeta - z|^2} \leq \frac{|\Delta z|}{(|\zeta - z| - |\Delta z|)|\zeta - z|^2}$$

$$\leq \frac{|\Delta z|}{(d - |\Delta z|)|\zeta - z|^2} \longrightarrow 0$$

as Δz goes to 0. Since f is continuous, it's bounded. So $\left| \frac{F(z') - F(z)}{z' - z} - \int_C \frac{f(\zeta}{(\zeta - z)^2} d\zeta \right|$ goes to 0 as Δz goes to 0. This shows F is analytic in a neighbourhood of z. Since z is arbitrarily chosen, we conclude F is analytic off of C. And since we only need the boundedness of f in the above proof, the conditions on f could be weakend.□

2.

Solution: This is sort of tedious computation. First of all, note in a neighbourhood of 0, the functions $(1 - z)^{-1}$, $\frac{1}{(1-z)^2}$ and $\log(1 - z)$ are all analytic, so by Theorem 6.1 on page 140, they are all uniformly approximated by their power series. So, you really don't have to prove the uniformness of the approximation. Once get rid of the theoretical part, the remaining treadmill is just calculus. A convincing way of doing it is to work by induction to find out the nth derivative of each function. Note the derivative of $\log(1 - z)$ is $(1 - z)^{-1}$ and the derivative of $(1 - z)^{-1}$ is $(1 - z)^{-2}$, so you only need to work on one of them.□

3.

Proof: Just follow the hint and work by induction. □

5.

Solution: We take C as the circle $|z| = r$. Suppose $P_n(z)$ is the sum of the first n terms of e^z's power series, i.e. $P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^{n-1}}{(n-1)!}$. Then (6.5) on page 141 gives the estimation $|e^z - P_n(z)| = |z^n \frac{1}{2\pi i} \int_C \frac{e^\zeta}{\zeta^n} \frac{d\zeta}{\zeta - z}|$, for z in a small neighbourhood inside C. On the circle C, e^z has the upperbound e^r. So,

$$|e^z - P_n(z)| \leq \frac{1}{2\pi} |z|^n \int_C e^r r^{-n} (r - |z|)^{-1} \times 2\pi r = \frac{e^r r}{r - |z|} (\frac{|z|}{r})^n$$

For $|z| \leq 1$, we let $r > 1$. The above reasoning goes through and we get $|e^z - P_n(z)| \leq \frac{e^r r}{r-1} (\frac{1}{r})^n$. Furthermore, let $r = 2$, we get the upper estimation $e^2 (\frac{1}{2})^{n-1}$. The inequality $2 \log_{10} e - (n-1) \log_{10} 2 \leq -20$ shows n should be at least 70.□

5.3.

Proof: $\forall \epsilon \in (0,1)$, we have

$$\frac{1}{\pi} \int_{\epsilon \leq |z| \leq 1} f(x+iy) dxdy = \frac{1}{\pi} \int_0^{2\pi} \int_\epsilon^1 f(z) r d\theta dr = \frac{1}{\pi} \int_\epsilon^1 dr \int_0^{2\pi} f(z) r d\theta$$

Since $\frac{dz}{z} = \frac{ire^{i\theta} d\theta}{re^{i\theta}} = id\theta$ when z is on the circle $\{|z| = r\}$ where $r > 0$, we have $\frac{1}{\pi} \int_\epsilon^1 dr \int_0^{2\pi} f(z) r d\theta = \frac{1}{\pi i} \int_\epsilon^1 r dr \int_{|z|=r} \frac{f(z)}{z} dz = 2 \int_\epsilon^1 f(0) r dr = f(0)(1 - \epsilon^2)$. Note

$$\frac{1}{\pi} \int_{|z| \leq 1} f(x+iy) dxdy = \frac{1}{\pi} \int_{\epsilon \leq |z| \leq 1} f(x+iy) dxdy + \frac{1}{\pi} \int_{|z| < \epsilon} f(x+iy) dxdy$$

The second term goes to 0 as ϵ goes to 0 while the first term goes to $f(0)$. So let $\epsilon \longrightarrow 0$, we are done. □