MATH 418 COMPLEX VARIABLES
Homework 6 Solution

Due March 6, 2001

Note: If you have any questions about the solution, or you think there are some typos/errors in the solution, please e-mail me. I'll double-check it and then reply to you. Thank you.

1. Solution: This kind of problem can be solved quite easily by looking at a function's Laurent series. Unfortunately, the most useful theorem is not in section 8, but in section 9 (Theorem 9.4, page 167). Of course, straightforward observation is also beneficial in some cases.

e^z: e^z is analytic everywhere in $\mathbb{C}$. To judge the property of ∞, we consider all the three possibilities. First, ∞ cannot be removable by problem 3, since e^z is not a constant function. Second, ∞ is not a pole, since $|e^{in}| \leq 1$, no matter how large $n \in \mathbb{N}$ is. So ∞ is an essential singularity.

$\frac{\cos z}{z}$: Only 0 or ∞ could have problems since $\frac{\cos z}{z}$ is analytic elwhere. Note the Laurent series of $\frac{\cos z}{z} = \sum_{n=2k}^{\infty} \frac{(iz)^n}{zn!}$, where $k \in \mathbb{N} \cup \{0\}$, we conclude 0 is a pole by Theorem 9.4, since $1/z$ appears in the series. To to see the property of ∞, replace z with $1/\zeta$, it's clear that $\zeta = 0$ is an essential singularity, by Theorem 9.4. Hence ∞ is an essential singularity of $\frac{\cos z}{z}$.

$\frac{e^z-1}{z(z-1)}$: 1 is a pole and 0 is removable. Replace z with $1/\zeta$, we get $(e^{\frac{1}{\zeta}} - 1)\frac{\zeta^2}{1-\zeta}$. It's clear that this function is not differenctiable at $\zeta = 0$(argue by direct computation according to the definition of being analytic). So, ∞ cannot be removable. Furthermore, if $z \longrightarrow \infty$ along the negative x-axis, then $\frac{e^z-1}{z(z-1)}$ goes to 0. So, ∞ cannot be a pole. Hence, ∞ has to be an essential singularity.

$\frac{z^2-1}{z^2+1}$: This function is equal to $1 - \frac{2}{z^2+1}$. So, i and $-i$ are two poles. Replace z with $1/\zeta$, we get $\frac{1-\zeta^2}{1+\zeta^2}$. This new function is differentiable at $\zeta = 0$. So, ∞ is a removable singularity of the original function.

$\frac{z^5}{z^3+z}$: By similar arguement, i and $-i$ are two poles.0 is a removable singularity. And ∞ is also a pole, since after replacing z with $1/\zeta$, we get $\frac{1}{\zeta^2+\zeta^4}$.

$e^{\cosh z}$: $\cosh z$ is an entire function, so is e^z. Since the function under consideration is the composition of two entire functions, it's entire. To judge ∞, note first by problem 3, ∞ cannot be removable. Let $z = in$ where n is just a natural number. Then we can see, as $n \longrightarrow +\infty$, henc $z \longrightarrow \infty$, $e^{\cosh z}$ is bounded. So, ∞ cannot be a pole. So, ∞ has to be an essential singularity.

$\frac{z(z-\pi)^2}{(\sin z)^2}$: We first solve the equation $e^{iz} = e^{-iz}$ and get solutions $z = k\pi$ where $k \in \mathbb{Z}$. For $k \neq 1, 0$, $k\pi$ becomes a pole since $\sin z = 0$ here. For 0, as $z \longrightarrow 0$, $\frac{\sin z}{z} \longrightarrow 1$, by the definition of the derivative of $\sin z$ at 0. So, 0 is a pole. For π, note $\sin(z - \pi) = -\sin z$, we again return to the previous case. But this time the dominator $\sin(z - \pi)$ and the nominator $(z - \pi)$ have the same power. So, $\frac{z(z-\pi)^2}{(\sin z)^2} \longrightarrow \pi$ as $z \longrightarrow \pi$. Hence, π is a removable singularity. Let $z \longrightarrow \infty$ along the positive x-axis, the function has no limit. So, ∞ cannot be a pole or removable. So, it's an essential singularity. □

2.
Solution: $f(z) = \frac{(z^2-1)(z-2)^3}{(\sin \pi z)^3}$. So we have the following equalities: $1/f(z) = \frac{(\sin \pi z)^3}{(z+1)(z-1)(z-2)^3} = -\frac{(\sin pi(z-1))^3}{(z+1)(z-1)(z-2)^3} = -\frac{(\sin \pi(z+1))^3}{(z+1)(z-1)(z-2)^3} = \frac{(\sin pi(z-2))^3}{(z+1)(z-1)(z-2)^3}$. So, it's clear that $1, -1, 2$ are removable singularities. Let $z = n$ where n is a natural number, then $1/f(z) = 0$ for any n. So, ∞ cannot be a pole. Let $z = -in$, then $1/f(z) \longrightarrow \infty$ as $n \longrightarrow \infty$. So, ∞ cannot be removable. Therefore ∞ has to be essential. □

3.
Proof: If a function is analytic in the extended plane, then in particular, it's analytic at ∞. So it must have a definite finite value at ∞ and is continuous at ∞. Hence, it is bounded in a neighbourhood of ∞, say, $\{z : |z| > M\}$ for some positive number M. Meanwhile, this function is bounded in the closed disc $\{z : |z| \leq M\}$. So, this analytic function is bounded on the whole plane. By Liouville's Theorem, it has to be a constant.□

4.
Proof: To find the principal part of the function at -1, we replace $z+1$ with ζ, and get

$$8z^3(z+1)^{-1}(z-1)^{-2} = 2\frac{(\zeta-1)^3}{\zeta(1-\zeta+\frac{\zeta^2}{4})} = 2\frac{\zeta^3-3\zeta^2+3\zeta-1}{\zeta}\sum_{k=0}^{\infty}(\zeta-\frac{\zeta^2}{4})^k$$

So, it's clear that the principal part is $-2/\zeta$. Replace ζ with $z+1$, we see the pricipal part of the original function at -1 is $\frac{-2}{z+1}$.
To find the pricipal part of the function at 1, we replace $z-1$ with ζ, and get

$$8z^3(z+1)^{-1}(z-1)^{-2} = \frac{8(\zeta+1)^3}{\zeta^2(\zeta+2)} = 4 \times \frac{\zeta^3+3\zeta^2+3\zeta+1}{\zeta^2(1+\frac{\zeta}{2})} = 4 \times \frac{\zeta^3+3\zeta^2+3\zeta+1}{\zeta^2}\sum_{k=0}^{\infty}(-\frac{\zeta}{2})^k$$

So the principla part is $4 \times [\frac{3\zeta+1}{\zeta^2} + \frac{1}{\zeta^2}(-\frac{\zeta}{2})] = \frac{4}{\zeta^2} + \frac{10}{\zeta}$. This is what we want. □

5.
Proof: Replace z with $1/\zeta$ in $(z^2+1)^2/(z^2-z)$, and take advantage of geometric series, we get $(z^2+1)^2/(z^2-z) = (\frac{1}{\zeta^2}+2+\zeta^2)\sum_{n=0}^{\infty}\zeta^n$. So, the principle part is $\frac{1}{\zeta^2}+\frac{1}{\zeta}$. Replace $1/\zeta$ with z, we get z^2+z. □

7.
Proof: $\forall r > 0, \theta \in \mathbb{R}$, to let $e^z = e^{x+iy} = re^{i\theta}$, we only need to let $x = \log r$ and $y = \theta$. Since $|e^z| = e^x > 0$, e^z cannot assume 0.
$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$. If we let $z = x+iy$ and let $a+bi$ be any complex number, we need to see if the equation $\sin z = a+bi$ has a solution in any neighbourhood of ∞. Indeed, this equation can be reduced to $(e^{-y}-e^y)\cos x = -2b$, and $(e^{-y}+e^y)\sin x = 2a$. This alwasy has solution evne if one of, or both of x, y goes to ∞(Actually, to complete this proof, you should solve this system of equations and find the explicit formula for a and b, in terms of x and y. Or you may want to calculate the Jacobian determinant of a, b with respect to x, y, i.e. $\frac{\partial(a,b)}{\partial(x,y)}$). □

9.

Proof:

$$(z-1)/(z+1) = \frac{1-1/z}{1+1/z} = (1-\frac{1}{z})(\sum_{k=0}^{\infty}(-1)^k/z^k) = 1+2\sum_{k=1}^{\infty}(-\frac{1}{z})^k$$

. □

1.

Proof: By definition (Theorem 9.2 page 164), Laurent coefficient

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^{j+1}}dz$$

In this problem, $\alpha = 0$. And

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{j+1}}dz = \sum_{k=-m}^{k=n}\frac{A_k}{2\pi i}\int_C \frac{z^k}{z^{j+1}}dz = \sum_{k=-m}^{k=n}\frac{A_k}{2\pi i}\int_C z^{k-j-1}dz$$

Note

$$\frac{1}{2\pi i}\int_C z^{k-j-1}dz = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{otherwise} \end{cases}$$

So, we conclude $a_j = A_j$ for $-m \leq j \leq n$ and $a_j = 0$ otherwise. □